(1) Find a recurrence relation for the number of different ways to hand out a piece of chewing gum (worth 1 cent) or a candy bar (worth 10 cents) or a donut (worth 20 cents), one item a day, until n cents worth of food has been given away.

\[ A_0 = 0 \]

\[ C_x \quad 1 \leq x \leq 9 = 1 \]

\[ C_x \quad 10 \leq x \leq 19 = 2 \]

\[ A_n = C_{n-1} + C_{n-10} + C_{n-20} \]

**Assumption:** Ordering of days matter

\[ C_x \quad x < 0 = 0 \]

\[ C_0 = 1 \]

\[ C_n = C_{n-1} + C_{n-10} + C_{n-20} \]

**At:** We can get to n cents by
- adding gum to a (n-1) cent purchase
- adding candy to a (n-10) cent purchase
- adding a donut to a (n-20) cent purchase.

We know the 3 ways to make these purchases are
\[ C_{n-1}, C_{n-10}, C_{n-20}, \text{ so let's add them up,} \]
(2) Solve the recurrence relation you found in problem (1).
\[ c_n = c_{n-1} + c_{n-10} + c_{n-20} \]

**Characteristic Equation:**
\[ x^{20} - x^{19} - x^{10} - 1 = 0. \]

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**Solve the following recurrence relation.**
\[ c_n = 5c_{n-1} - 8c_{n-2} + 4c_{n-3} \quad c_0 = c_1 = c_2 = 1 \]

**Characteristic Equation**
\[ x^3 - 5x^2 + 8x - 4 = 0 = (x-2)^2(x-1) \]
\[ q_1 = 2 \quad (2\text{nd order}) \]
\[ q_2 = 1 \]
\[ c_n = a_1 (q_1)^n + a_2 n (q_2) + a_3 (q_2)^n = a_1 (2)^n + a_2 n (2)^n + a_3 (1)^n \]
\[ c_0 = a_1 (1) + 0 a_2 + a_3 = 1 \]
\[ c_1 = a_1 (2) + a_2 (1) 2 + a_3 = 1 \]
\[ c_2 = a_1 (4) + a_2 (2) (4) + a_3 = 1 \]

**Solve using linear algebra.**
(3) By using generating series, determine the number of integer solutions to the composition problem \( e_1 + e_2 + e_3 = 22 \) subject to \( 3 \leq e_1 \leq 8, \ 6 \leq e_2 \leq 10 \) and \( 2 \leq e_3 \leq 7 \).

\[
\left( \sum_{i=1}^{8} x^i \right) \left( \sum_{j=1}^{10} x^j \right) \left( \sum_{k=2}^{7} x^k \right)
\]

\[
= \frac{\left( \sum_{i=2}^{9} x^i \right) \left( \sum_{j=2}^{12} x^j \right) \left( \sum_{k=2}^{9} x^k \right)}{\left( \sum_{i=1}^{8} x^i \right) \left( \sum_{j=1}^{10} x^j \right) \left( \sum_{k=2}^{7} x^k \right)}
\]

\[
= \sum \left( \binom{9+2}{9} x^9 \right)
\]

\[
= \sum \left( \binom{11+2}{11} x^{11} - x^{11} + x^{17} + x^{22} + x^{22} + x^{23} - x^{38} \right)
\]

\[
= \sum \left( \binom{11+2}{11} x^{11} - 2 x^{17} + 2 x^{22} - x^{23} \right)
\]

\[
\binom{11}{3} - \binom{8}{3} - 2 \binom{7}{3} + 2
\]
Using exponential generating series, determine the number of ways to put 20 people into two rooms $A$ and $B$ subject to the condition that room $A$ has at least three people in it. (You need to actually extract the desired coefficient from your generating series.)

Assume the people are distinguishable. The generating series for the number of ways to place $n$ people into 2 rooms where room $A$ has at least 3 people:

$$g(x) = \left( \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^{20}}{20!} \right)(1 + \frac{x^1}{2!} + \frac{x^2}{3!} + \ldots + \frac{x^{17}}{17!} + \ldots)$$

Choosing terms from the above equation to form a $\frac{x^{20}}{20!}$ term:

$$\frac{x^{20}}{20!} \cdot \left( \frac{x^2}{3!} \right)(\frac{x^{17}}{17!}) + \left( \frac{x^4}{4!} \right)(\frac{x^{15}}{16!}) + \ldots + \left( \frac{x^{20}}{20!} \right)(1)$$

$$x^{20} = 20! \cdot \sum_{\substack{i+j=20, \\
i \geq 3, \\
j \geq 0}} \frac{1}{i!j!}$$

So the total number of ways is

$$\frac{20!}{2!} \cdot \sum_{i=3}^{20} \frac{1}{i!(20-i)!}$$
(5) Give, with proof, a bijection between Dyck paths from \((0,0)\) to \((2n,0)\) and the set of standard Young tableaux of shape \((n,n)\), i.e., fillings of the \(2 \times n\) chessboard with numbers \(1,2,\ldots,2n\) such that the rows and columns strictly increase. For example, when \(n = 3\) we have five standard Young tableaux, namely:

\[
\begin{array}{cccc}
1 & 2 & 3 & \\
4 & 5 & 6 & \\
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 4 & \\
3 & 5 & 6 & \\
\end{array}
\quad
\begin{array}{cccc}
1 & 2 & 5 & \\
3 & 4 & 6 & \\
\end{array}
\quad
\begin{array}{cccc}
1 & 3 & 5 & \\
2 & 4 & 6 & \\
\end{array}
\quad
\begin{array}{cccc}
1 & 3 & 4 & \\
2 & 5 & 6 & \\
\end{array}
\]

\(\uparrow\) represents adding in the top row

\(\downarrow\) represents adding in the bottom row

The number represents the order added.

Because there must always be a smaller element to the left and up in a Young tableau, this reflects that those elements had to be formed earlier during the construction from a Dyck path, so at all stages \(\#T \geq \#J\).

Because the final tableau has \(n\) in the top row and \(n\) in the bottom row, total \(T = \text{total } J = n\), so Dyck path has length \(2n\).

**Example:**

\[
\begin{array}{cccc}
\downarrow & \uparrow & \uparrow & \uparrow \\
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
\Rightarrow
\begin{array}{cccc}
1 & 2 & 5 & \\
3 & 4 & 6 & \\
\end{array}
\]