(1) Use the binomial theorem to evaluate

\[ \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k}. \]

Integrate binomial formula

\[ \int_{0}^{1} \sum_{k=0}^{n} \binom{n}{k} x^k \, dx = \int_{0}^{1} (1+x)^n \, dx \]

\[ \sum_{k=0}^{n} \binom{n}{k} \int_{0}^{1} x^k \, dx = \frac{2^{n+1} - 1}{n+1} \quad (x=1) \]

\[ \sum_{n=0}^{\infty} \frac{1}{k+1} \binom{n}{k} \]
(2) Use the binomial theorem and the fact that

\[(1 + x)^a(1 + x)^b = (1 + x)^{a+b}\]

to prove the identity

\[\sum_{k=0}^{n} \binom{a}{k} \binom{b}{n-k} = \binom{a+b}{n}.\]

(Do not give a combinatorial proof.)

\[\binom{a}{k} x^k \quad \text{and} \quad \binom{b}{k} x^k\]

\[\sum_{k=0}^{n} \binom{a}{k} \binom{b}{n-k} = \binom{a+b}{n}\]

\[x^n \text{ coefficient of } (1 + x)^a(1 + x)^b = \sum_{k=0}^{\min(a,b)} \binom{a}{k} \binom{b}{n-k} = \binom{a+b}{n} \]

So

\[\sum_{k=0}^{n} \binom{a}{k} \binom{b}{n-k} = \binom{a+b}{n} \]
(3) Given \( n \) chemists and \( n \) biologists (assume that each person is either a chemist or a biologist, but not both), use inclusion-exclusion to obtain a summation formula for the number of ways to pair the \( 2n \) people as lab partners such that the \( i \)-th tallest chemist is not matched to the \( i \)-th tallest biologist. (Write the expression out.)

\[
S = \{ \text{all pairings } (i_1, 1), (i_2, 2), \ldots, (i_n, n) \}
\]

\[
A_k = \{ \text{pairings with } i_k = k \}
\]

\[
|A_k| = (n-1)!
\]

\[
|A_{k_1} \cap A_{k_2}| = (n-2)!
\]

\[
|A_{k_1} \cap A_{k_2} \cap \ldots \cap A_{k_\ell}| = (n-\ell)!
\]

\[
\sum_{\ell = 1}^{n} \sum_{k_1, \ldots, k_\ell \in \{1, \ldots, n\}} \frac{|A_{k_1} \cap \ldots \cap A_{k_\ell}|}{(-1)^{\ell}}
\]

\[
\sum_{\ell = 0}^{n} \binom{n}{\ell} (n-\ell)! \left(\frac{(-1)^{\ell}}{(n-\ell)!}\right)
\]
(4) Find the number of integral solutions of
\[ x_1 + x_2 + x_3 + x_4 = 17, \]
subject to \(0 \leq x_1 \leq 5, 2 \leq x_2 \leq 6, 0 \leq x_3 \leq 4,\) and \(0 \leq x_4.\)

\[ y_1 = x_2 - 2 \quad \quad y_1 + y_2 + y_3 + y_4 = 15 \]
\[ 0 \leq y_1 \leq 5 \quad \quad 0 \leq y_3 \leq 4 \]
\[ 0 \leq y_2 \leq 4 \quad \quad 0 \leq y_4 \]

\[ A_1 : y_1 > 6 \quad \quad A_2 : y_3 > 5 \]
\[ A_2 : y_2 > 5 \]

\[ |A_1 \cap A_2 \cap A_3| = \binom{18}{3} - \binom{12}{3} - \binom{13}{3} - \binom{13}{3} + \binom{7}{3} + \binom{7}{2} + \binom{8}{3} - 0 \]
(5) Use inclusion-exclusion to prove that

$$
\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^n = n!.
$$

$$\frac{n!}{k! (n-k)!}, (n-k)^n$$

$$\binom{n}{0} n^n - \binom{n}{1} (n-1)^n + \binom{n}{2} (n-2)^n$$

$$S = \{1, 2, \ldots, n\}$$

$$X = \{\text{sequences of } S\}$$

$$A_i = \{\text{sequences of } S - \{i\}\}$$

$$|X| = n^n$$

$$|A_i| = (n-1)^n$$

$$|A_i \cap A_j| = (n-2)^n$$

$$\vdots$$

$$|A_1 \cap A_2 \cap \ldots \cap A_n| = 0$$

# of \( A_i \) = \binom{n}{i}

# of \( A_i \cap A_j \) = \binom{2}{1}

\vdots

# of \( A_i \cap A_2 \cap \ldots \cap A_n \) = \binom{n}{n}

\( n! \) is \# permutations of \( S \). LOW, \# sequence w/ distinct numbers. This can only happen if there are no duplicates. To have a duplicate, must exclude another \#.