(1) Determine the number of integral solutions to

\[ x_1 + x_2 + x_3 + x_4 + x_5 = 2015 \]

subject to

\[ x_1 \geq 100, x_2 \geq 30, x_3 \geq -20, x_4 \geq -5, x_5 \geq -10. \]

\[ y_1 = x_1 - 100, \quad y_2 = x_2 - 30, \quad y_3 = x_3 + 20, \quad y_4 = x_4 + 5, \quad y_5 = x_5 + 10 \]

\[ \Rightarrow \quad y_1 + y_2 + y_3 + y_4 + y_5 = 1920 \]

\[ y_1, y_2, y_3, y_4, y_5 \geq 0 \]

\[ \Rightarrow \quad \binom{1920 + 5 - 1}{5 - 1} = \binom{1924}{4} \]
(2) Show that any set of 76 positive integers \( \leq 100 \) must contain 4 consecutive integers.

Use Pigeonhole.

\[
\begin{align*}
\{ &1, 2, 3, 4 \\
&5, 6, 7, 8 \\
&9, 10, 11, 12 \}
\end{align*}
\]

\[
\{ &\ldots \\
&97, 98, 99, 100 \}
\]

\[
\begin{align*}
n &= 25 \\
c &= 4
\end{align*}
\]

\[
\begin{align*}
r &= n + 1 \\
(25)4 &= 25 + 1 \\
100 &= 25 + 1 \\
&= 76
\end{align*}
\]

\[
\exists \text{ a group with at least 4 objects by strong Pigeonhole.}
\]
(3) A footrace takes place among four players. If ties are allowed (even all four players finish the same time), how many ways are there for the race to finish?

Total = 36 + 8 + 1 + 24 + 12 = 81

2 of the 4 tie
- # ways to choose the tied pair
- $\binom{4}{2}$
  - perms of each pair 1st, 2nd, 3rd
  - $\implies 3!$
  - $\implies 3!(\frac{4}{2}) = 36$

3 of the 4 tie
- # ways to choose triple
- $\binom{4}{3}$
  - triple ties either 1st or 2nd
  - $\implies 2!$
  - $\implies 2!(\frac{4}{3}) = 8$

All 4 tie $\implies 1$
No ties $\implies 4! = 24$
2 pairs of ties
- # ways to choose tied pair $\binom{4}{2}$
  - perms of each pair 1st, 2nd
  - $\implies 2! = \frac{4!}{2!} = 12$
(4) Prove the following binomial identity COMBINATORIALLY

\[ \sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}. \]

Left: In total we have \( n+r+1 \) elements. Since \( \binom{n+r+1}{r} = \binom{n+1}{n+1} \), it's choosing \( n+1 \) elements out of \( n+r+1 \) elements. \( \Rightarrow \binom{n+r+1}{n+1} \).

Right: Pick the largest element \( \max \) out of the \( n+1 \) elements first.

Then pick the rest \( n \) elements from those elements smaller than \( \max \). There are \( r+1 \) categories.

1. The largest is the \((n+1)\)th greatest. \( \Rightarrow \binom{n}{n} \) choose \( n \) elements from rest \( n \) elements.

2. The largest is the \((n+2)\)th greatest \( \Rightarrow \binom{n+1}{n} \), choose \( n \) elements from rest \((n+1)\) elements.

\[ \vdots \]

5. The largest is \((n+r)\) \( \Rightarrow \binom{n+r}{n} \) choose \( n \) elements from rest \((n+r)\) elements.

\[ \Rightarrow \sum_{k=0}^{r} \binom{n+k}{n} = \sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{n+1}. \]
(5) How many arrangements of the letters a, e, i, o, u, x, x, x, x, x, x, x, x (8 x's) are there if no two vowels can be consecutive?

Since the x's are identical, they can be placed in 1 way.

\[ \_ x \_ x \_ x \_ x \_ x \_ x \_ x \_ x \_ x \_ x \_ \]

9 places for vowels. Pick places for vowels \( \binom{9}{5} \) ways

Then permute vowels \( 5! \) ways.

Total arrangements: \( \binom{9}{5}5! = 15120 \)