This homework is due March 30 at noon. All answers must be fully justified. You must list all your sources and the names of the people you collaborated with.

1. Give combinatorial proofs for the following relations the Stirling numbers of the second kind satisfy:
   (a) \( S(n, 1) = 1 \) for \( n \geq 1 \)
   (b) \( S(n, 2) = 2^{n-1} - 1 \) for \( n \geq 2 \)
   (c) \( S(n, n - 1) = \binom{n}{2} \) for \( n \geq 1 \)
   (d) \( S(n, n - 2) = \binom{n}{3} + 3\binom{n}{4} \) for \( n \geq 2 \)

2. Fix \( n \), for which type \( c = (c_1, c_2, \ldots, c_n) \) do we have the smallest number of permutations of \( \{1, 2, \ldots, n\} \) of this type?

3. Prove that the exponential generating function for \( S(n, k) \) with \( k \) fixed is
   \[
   \sum_{n=k}^{\infty} S(n, k) \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!}.
   \]

4. A sequence of positive integers is dull if for any \( k > 1 \) which appears in the sequence, the number \( k - 1 \) appears at least once before the first occurrence of \( k \). Find the number of dull sequences of length \( n \) where the largest number is \( m \).

5. Consider the generating function for the Stirling numbers of the second kind \( S(n, k) \) where \( n \) is fixed and \( k \) varies: \( p_n(x) = \sum_{k=0}^{\infty} S(n, k) x^k \). Prove that \( p_0(x), p_1(x), p_2(x), \ldots \) satisfies \( p_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y) \).

6. For each of the following permutations, compute all its cycles and cycle type. Recall that the permutation \( i_1, i_2, \ldots, i_n \) corresponds to the function \( f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\} \) defined by \( f(j) = i_j \).
   (a) 3, 4, 5, 6, 7, 8, 9, 1, 2
   (b) 10, 9, 1, 8, 4, 5, 6, 3, 2, 7
   (c) 2, 4, 9, 1, 3, 5, 8, 7, 6, 10