This homework is due March 2 at noon. All answers must be fully justified. You must list all your sources and the names of the people you collaborated with.

1. Prove the identity (in the form given) combinatorially:

\[
\binom{n}{k} - \binom{n-3}{k} = \binom{n-1}{k-1} - \binom{n-2}{k-1} + \binom{n-3}{k-1}.
\]

(Hint: Let \(S\) be a set with three distinguished elements \(a, b,\) and \(c\) and count certain \(k\)-subsets of \(S\).)

2. By integrating the binomial expansion, prove that, for a positive integer \(n\),

\[
1 + \frac{1}{2} \binom{n}{1} + \frac{1}{3} \binom{n}{2} + \cdots + \frac{1}{n+1} \binom{n}{n} = \frac{2^{n+1} - 1}{n+1}.
\]

3. Let \(n\) and \(k\) be positive integers. Give a combinatorial proof that

\[
\sum_{k=1}^{n} k \binom{n}{k}^2 = n \binom{2n-1}{n-1}.
\]

4. Prove that

\[
\sum_{n_1, n_2, n_3 \geq 0, n_1 + n_2 + n_3 = n} \frac{n!}{n_1!n_2!n_3!}(-1)^{n_1 - n_2 + n_3} = (-3)^n.
\]

5. Find and prove a formula for

\[
\sum_{r+s+t=n} \binom{m_1}{r} \binom{m_2}{s} \binom{m_3}{t}.
\]

6. Prove that the only antichain of \(S = \{1, 2, 3, 4\}\) of size 6 is the antichain of all 2-subsets of \(S\).

7. A talk show host has just bought 10 new jokes. Each night she tells some of the jokes. What is the largest number of nights on which you can tune in so that you never hear on one night at least all the jokes you heard on one of the other nights? (Thus, for instance, it is acceptable that you hear jokes 1, 2, and 3 on one night, jokes 3 and 4 on another, and jokes 1, 2, and 4 on a third. It is not acceptable that you hear jokes 1 and 2 on one night and joke 2 on another night.)