This homework is due March 30 at noon. All answers must be fully justified. You must list all your sources and the names of the people you collaborated with.

1. For each integer $n > 2$, determine a self-conjugate partition of $n$ that has at least two parts.

2. Prove that the number of partitions of $n$ in which no part appears exactly once is equal to the number of partitions of $n$ with no parts congruent to 1 or 5 (mod 6).

3. Prove that the partition function $p(n)$ (number of partitions of $n$) satisfies $p(n + 1) > p(n)$.

4. By considering partitions with distinct (that is, non-repeated) parts, prove that

\[
\prod_{k \geq 1} (1 + x^k) = 1 + \sum_{m \geq 1} \left( \frac{x^{m(m+1)/2}}{(\prod_{k=1}^{m} (1 - x^k))} \right)
\]

(Hint: look for a "maximal triangle" rather than a Durfee square.)

5. Find the exponential generating series, and identify the appropriate coefficient, for the number of ways to deal a sequence of 13 cards (from a standard 52 card deck) if the suits are ignored and only the values of the cards are noted.

6. Let $n \geq 1$, and let $f(n)$ be the number of partitions of $n$ such that for all $k$, the part $k$ occurs at most $k$ times. Let $g(n)$ be the number of partitions of $n$ such that no part has the form $i(i + 1)$ (i.e., no parts equal to 2, 6, 12, 20, ...). Show that $f(n) = g(n)$.

7. Find a bijection between partitions $\lambda \vdash n$ with Durfee square of side-length $r$ and integer arrays

$$A_\lambda = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$$

such that $a_1 > a_2 > \cdots > a_r \geq 0$, $b_1 > b_2 > \cdots > b_r \geq 0$, and $r + \sum (a_i + b_i) = n$. 