A combinatorial approach to the study of divisors on $\overline{M}_{0,n}$

Laura Escobar

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Goal

Goal: Illustrate how a problem from algebraic geometry can be approached using combinatorics

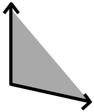
Outline

Cones

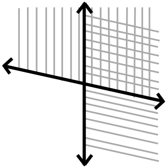
Definition

A cone is the positive span of a finite number of vectors, i.e., a set of the form

$$\text{pos}(v_1, \ldots, v_k) := \{\lambda_1 v_1 + \cdots \lambda_k v_k : \lambda_i \geq 0\}$$



Cones can also be expressed as a finite intersection of halfspaces.

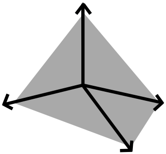


Fans

Definition

A fan is a family of nonempty cones such that

1. Every nonempty face of a cone in the fan is also a cone of the fan,
2. the intersection of any two cones is a face of both.



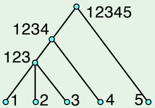
Important example: Space of Phylogenetic trees

Definition

- A rooted tree is a graph that has no cycles and which has a vertex of degree at least 2 labelled as the root of the tree.
- The leaves of the tree are all the vertices of degree 1; we label them from 1 to n.

Each vertex of the tree corresponds to a subset of $\{1, \ldots, n\}$

Example

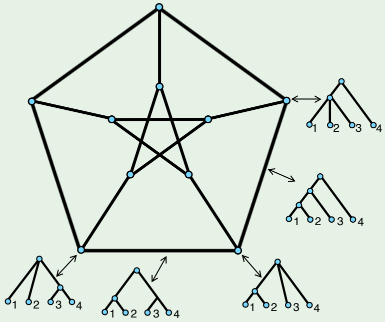


Coarse subdivision on $\mathcal{T}(K_n)$

- There is a fan whose cones are in 1-1 correspondence with rooted trees with n labelled leaves.
- Maximal cones correspond to binary trees.
- Rays correspond to subsets of $\{1, \ldots, n\}$ of size ≥ 2, so a cone corresponding to the tree T is generated by the rays corresponding to the vertices of T.
- The union of the cones of this fan is the space of phylogenetic trees

Example

n=4



Star[1]-convex functions

Definition

Given a fan Δ, $N(\Delta)$ is the set of piecewise linear functions $\varphi : \bigcup_{\sigma\in\Delta} \sigma \to \mathbb{R}$ that are linear on each cone of Δ.

$N(\Delta)$ is isomorphic to $\mathbb{R}^{\#\text{ of rays}}$, i.e., a function φ is determined by its values on the rays.

Phylogenetic case

A function $\varphi \in N(\Delta)$ is determined by the values on the rays v_I where I is a subset of $\{1, \ldots, n\}$ of size ≥ 2.

Definition

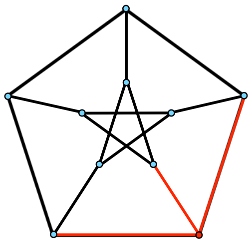
Let $\sigma \in \Delta$, we say that $\varphi \in N(\Delta)$ is star^1-convex on σ if it satisfies that

$$\varphi\,(u_1 + \cdots + u_k) \leq \varphi(u_1) + \cdots \varphi(u_k)$$

for each $u_1, \ldots, u_k$ such that

1. $u_1 + \cdots + u_k \in \sigma$, and
2. each $u_i \in \tau_i$ where $\tau_i \supset \sigma$ and $\dim(\tau_i) = \dim(\sigma) + 1$, i.e., $\tau_i \in \text{star}^1(\sigma)$.

Example



star$^1(\sigma)$ of the cone σ corresponding to the red vertex.

Cones on $N(\Delta)$

Definition

Let σ be a cone of a fan Δ, define

- $\mathfrak{C}(\sigma)$, the set of functions $\varphi \in N(\Delta)$ that are star[1]-convex on σ,
- the set of functions in $N(\Delta)$ star[1] convex on all cones $\sigma \in \Delta$:
  $\mathfrak{L}(\Delta) := \bigcap_{\sigma \in \Delta} \mathfrak{C}(\sigma)$, and
- the set of functions in $N(\Delta)$ star[1] convex on all cones $\sigma \in \Delta$ of codimension 1:
  $\mathfrak{U}(\Delta) := \bigcap_{\sigma \in \Delta,\ \mathrm{codim}(\sigma)=1} \mathfrak{C}(\sigma)$.

Question

Clearly $\mathfrak{L}(\Delta) \subseteq \mathfrak{U}(\Delta)$, but are the two cones equal for certain fans?

Outline

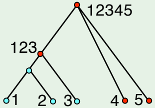
The cone $\mathfrak{U}$

Theorem

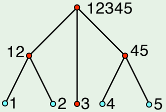
Trees corresponding to cones of codimension 1 have only one vertex with exactly 3 children. Each of these trees gives a halfspace for $\mathfrak{U}$ which depends only on this vertex and its children.

Example

For K_5, $\mathfrak{U}$ is the intersection of 65 halfspaces in $\mathbb{R}^{26}$. Some of the halfspaces:



$$\varphi(123) + \varphi(4) + \varphi(5) + \varphi(12345) \leq \varphi(1234) + \varphi(1235) + \varphi(45)$$



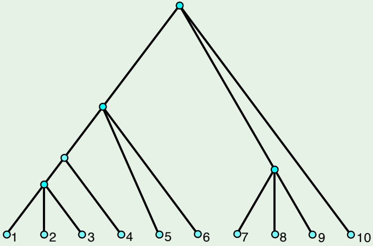
$$\varphi(12) + \varphi(3) + \varphi(45) + \varphi(12345) \leq \varphi(123) + \varphi(345) + \varphi(1245)$$

The cone $\mathfrak{L}$

Theorem

Let T be the tree corresponding to the cone σ and $r_1, \ldots, r_k$ be all the vertices of T having ≥ 3 children. Then computing cone $\mathfrak{C}(\sigma)$ can be reduced to computing smaller cones $\mathfrak{C}(r_i)$ where each such cone depends only on vertex r_i and its children.

Example



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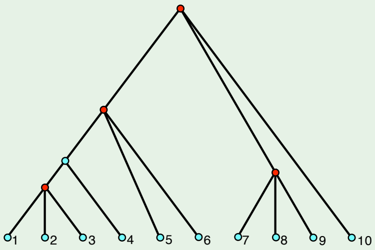

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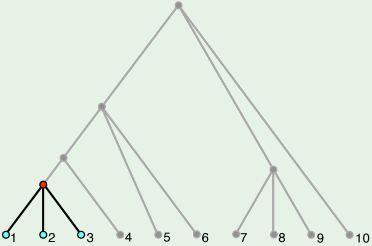

$$\varphi(1) + \varphi(2) + \varphi(3) + \varphi(123) \leq \varphi(12) + \varphi(13) + \varphi(23)$$

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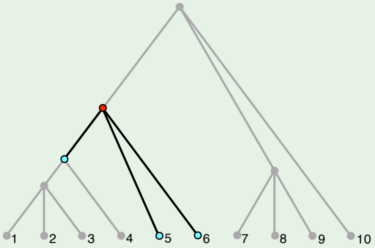

$$\varphi(1234) + \varphi(5) + \varphi(6) + \varphi(123456) \leq \varphi(12345) + \varphi(12346) + \varphi(56)$$

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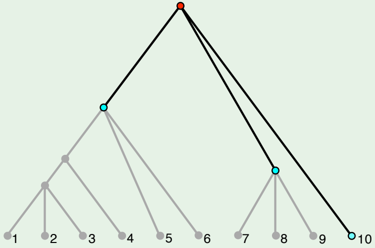

$$\varphi(123456) + \varphi(789) + \varphi(10) + \varphi(12345678910) \leq \varphi(123456789) + \varphi(12345610) + \varphi(78910)$$

The cone $\mathfrak{L}$

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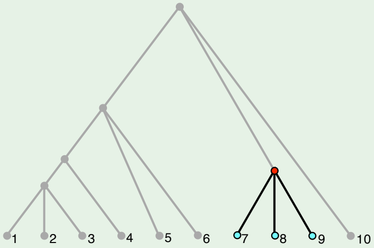

$$\varphi(7) + \varphi(8) + \varphi(9) + \varphi(789) \leq \varphi(78) + \varphi(79) + \varphi(89)$$

Inductive approach

Theorem

If $\mathfrak{L} = \mathfrak{U}$ for the space of phylogenetic trees with $n-1$ leaves and $\mathfrak{U}$ is contained in the intersection of the cones given by the trees with only one internal vertex and at most n leaves, then $\mathfrak{L} = \mathfrak{U}$ for the space of phylogenetic trees with n leaves.

Outline

1. The combinatorial Problem
   - The space
   - The players
   - The game
2. The cones $\mathfrak{U}$ and $\mathfrak{L}$ for the space of phylogenetic trees
   - The cone $\mathfrak{U}$
   - The cone $\mathfrak{L}$
3. The algebraic geometry story
   - Moduli spaces
   - Divisors
   - Useful tool

General idea

Theorem (Gibney and Maclagan)

The cones $\mathfrak{L}$ and $\mathfrak{U}$ give us a tool to compute an important cone which arises in algebraic geometry.

- A central goal in algebraic geometry is to understand maps $X \longrightarrow \mathbb{P}^k$, for a projective variety X.
- A main tool in studying these maps is the nef cone of X.
- Interesting unknown case when $X = \overline{M}_{0,n}$.

Moduli spaces

Definition

The moduli space $M_{0,n}$ is a geometric space whose points correspond to isomorphism classes of smooth curves of genus 0 with n distinct marked points.

- $M_{0,n} = \{\mathbb{P}^1$ with n distinct marked points$\}/$ automorphisms.
- Smooth space of dimension $n - 3$.
- Understanding of this space tells us a lot about curves.

Deligne-Mumford compactification $\overline{M}_{0,n}$

Add every curve with n marked points whose group of automorphisms fixing those points is finite.

Divisors (simplified)

Definition

A Divisor of a variety X is a finite sum of the form $\sum_i a_i D_i$ where each $a_i \in \mathbb{R}$ and each D_i is a codimension 1 subvariety of X.

Definition

- Let D be a divisor and C a curve in X, then $D \cdot C := \sum_i a_i \, |D_i \cap C|$.
- The nef cone of X is the cone generated by divisors such that $D \cdot C \geq 0$ for all curves C.

Example

$D = \sum_i a_i D_i$ with $a_i \geq 0$ is in the nef cone.

Useful tool

There is a natural embedding of $\overline{M}_{0,n} \hookrightarrow X_\Delta$, where X_Δ is the toric variety of the fan of the space of phylogenetic trees.

The cones $\mathfrak{L}(\Delta)$ and $\mathfrak{U}(\Delta)$ are cones of divisors on X_Δ.

Gibney and Maclagan use this embedding to pull back these cones to cones of divisors on $\overline{M}_{0,n}$ which give upper and lower bounds for the nef cone of $\overline{M}_{0,n}$.

If we can prove $\mathfrak{L}(\Delta_n) = \mathfrak{U}(\Delta_n)$, where Δ_n is the space of phylogenetic trees with n leaves, then we would have a nice description of the nef cone of $\overline{M}_{0,n}$, which is in general hard to compute.

This technique can also be applied to other projective varieties X for which there is a nice embedding to a toric variety.

Thank you!