A combinatorial approach to the study of divisors on $\overline{M}_{0,n}$

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Goal

**Goal:** Illustrate how a problem from algebraic geometry can be approached using combinatorics.
1 The combinatorial Problem
   - The space
   - The players
   - The game

2 The cones $\mathcal{U}$ and $\mathcal{L}$ for the space of phylogenetic trees
   - The cone $\mathcal{U}$
   - The cone $\mathcal{L}$

3 The algebraic geometry story
   - Moduli spaces
   - Divisors
   - Useful tool
Outline

1. The combinatorial Problem
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Cones

**Definition**

A cone is the positive span of a finite number of vectors, i.e., a set of the form

$$\text{pos}(v_1, \ldots, v_k) := \{\lambda_1 v_1 + \cdots + \lambda_k v_k : \lambda_i \geq 0\}$$

Cones can also be expressed as a finite intersection of halfspaces.
Fans

Definition

A fan is a family of nonempty cones such that

1. Every nonempty face of a cone in the fan is also a cone of the fan,
2. the intersection of any two cones is a face of both.
Important example: Space of Phylogenetic trees

Definition

- A rooted tree is a graph that has no cycles and which has a vertex of degree at least 2 labelled as the root of the tree.
- The leaves of the tree are all the vertices of degree 1; we label them from 1 to \( n \).

Each vertex of the tree corresponds to a subset of \( \{1, \ldots, n\} \)

Example

```
1 2 3 4 5
123
1234
12345
```

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Combinatorics of nef \( \overline{M_{0,n}} \)
Coarse subdivision on $\mathcal{T}(K_n)$

- There is a fan whose cones are in 1-1 correspondence with rooted trees with $n$ labelled leaves.
- Maximal cones correspond to binary trees.
- Rays correspond to subsets of $\{1, \ldots, n\}$ of size $\geq 2$, so a cone corresponding to the tree $T$ is generated by the rays corresponding to the vertices of $T$.
- The union of the cones of this fan is the space of phylogenetic trees.

Example

$n=4$

[Diagram showing the coarse subdivision on $\mathcal{T}(K_n)$]
Star\(^1\)-convex functions

**Definition**

Given a fan \(\Delta\), \(N(\Delta)\) is the set of piecewise linear functions \(\varphi : \bigcup_{\sigma \in \Delta} \sigma \rightarrow \mathbb{R}\) that are linear on each cone of \(\Delta\).

\(N(\Delta)\) is isomorphic to \(\mathbb{R}^\#\) of rays, i.e., a function \(\varphi\) is determined by its values on the rays.

**Phylogenetic case**

A function \(\varphi \in N(\Delta)\) is determined by the values on the rays \(v_I\) where \(I\) is a subset of \(\{1, \ldots, n\}\) of size \(\geq 2\).
Definition

Let \( \sigma \in \Delta \), we say that \( \varphi \in N(\Delta) \) is \( \text{star}^1 \)-convex on \( \sigma \) if it satisfies that

\[
\varphi(u_1 + \cdots + u_k) \leq \varphi(u_1) + \cdots + \varphi(u_k)
\]

for each \( u_1, \ldots, u_k \) such that

1. \( u_1 + \cdots + u_k \in \sigma \), and
2. each \( u_i \in \tau_i \) where \( \tau_i \supset \sigma \) and \( \dim(\tau_i) = \dim(\sigma) + 1 \), i.e., \( \tau_i \in \text{star}^1(\sigma) \).

Example

star\(^1\)(\( \sigma \)) of the cone \( \sigma \) corresponding to the red vertex.
Cones on $N(\Delta)$

**Definition**

Let $\sigma$ be a cone of a fan $\Delta$, define

- $\mathcal{C}(\sigma)$, the set of functions $\varphi \in N(\Delta)$ that are star$^1$-convex on $\sigma$,
- the set of functions in $N(\Delta)$ star$^1$ convex on all cones $\sigma \in \Delta$:
  $$\mathcal{L}(\Delta) := \bigcap_{\sigma \in \Delta} \mathcal{C}(\sigma),$$
- the set of functions in $N(\Delta)$ star$^1$ convex on all cones $\sigma \in \Delta$ of codimension 1:
  $$\mathcal{U}(\Delta) := \bigcap_{\sigma \in \Delta, \text{codim}(\sigma)=1} \mathcal{C}(\sigma).$$

**Question**

Clearly $\mathcal{L}(\Delta) \subseteq \mathcal{U}(\Delta)$, but are the two cones equal for certain fans?
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The combinatorial Problem
The cones $\mathcal{U}$ and $\mathcal{L}$ for the space of phylogenetic trees
The algebraic geometry story

The cone $\mathcal{U}$

**Theorem**

Trees corresponding to cones of codimension 1 have only one vertex with exactly 3 children. Each of these trees gives a halfspace for $\mathcal{U}$ which depends only on this vertex and its children.

**Example**

For $K_5$, $\mathcal{U}$ is the intersection of 65 halfspaces in $\mathbb{R}^{26}$. Some of the halfspaces:

\[
\varphi(123) + \varphi(4) + \varphi(5) + \varphi(12345) \\
\leq \varphi(1234) + \varphi(1235) + \varphi(45)
\]

\[
\varphi(12) + \varphi(3) + \varphi(45) + \varphi(12345) \\
\leq \varphi(123) + \varphi(345) + \varphi(1245)
\]
The cone $\mathcal{L}$

**Theorem**

Let $T$ be the tree corresponding to the cone $\sigma$ and $r_1, \ldots, r_k$ be all the vertices of $T$ having $\geq 3$ children. Then computing cone $\mathcal{C}(\sigma)$ can be reduced to computing smaller cones $\mathcal{C}(r_i)$ where each such cone depends only on vertex $r_i$ and its children.

**Example**

![Diagram of a tree with vertices numbered 1 to 10, illustrating the cone structure.](image-url)
The cone $\mathcal{L}$

**Theorem**

Let $T$ be the tree corresponding to the cone $\sigma$ and $r_1, \ldots, r_k$ be all the vertices of $T$ having $\geq 3$ children. Then computing cone $\mathcal{C}(\sigma)$ can be reduced to computing smaller cones $\mathcal{C}(r_i)$ where each such cone depends only on vertex $r_i$ and its children.

**Example**

![Diagram of a tree with vertices labeled 1 to 10, illustrating the cone $\mathcal{L}$ with specific vertices and connections for the theorem statement.]
The cone $\mathcal{L}$

**Theorem**

Let $T$ be the tree corresponding to the cone $\sigma$ and $r_1, \ldots, r_k$ be all the vertices of $T$ having $\geq 3$ children. Then computing cone $\mathcal{L}(\sigma)$ can be reduced to computing smaller cones $\mathcal{L}(r_i)$ where each such cone depends only on vertex $r_i$ and its children.

**Example**

$$\varphi(1) + \varphi(2) + \varphi(3) + \varphi(123) \leq \varphi(12) + \varphi(13) + \varphi(23)$$
The cone $\mathcal{L}$

Theorem

Let $T$ be the tree corresponding to the cone $\sigma$ and $r_1, \ldots, r_k$ be all the vertices of $T$ having $\geq 3$ children. Then computing cone $\mathcal{L}(\sigma)$ can be reduced to computing smaller cones $\mathcal{L}(r_i)$ where each such cone depends only on vertex $r_i$ and its children.

Example

\[ \varphi(1234) + \varphi(5) + \varphi(6) + \varphi(123456) \leq \varphi(12345) + \varphi(12346) + \varphi(56) \]
The cone $\mathcal{L}$

**Theorem**

Let $T$ be the tree corresponding to the cone $\sigma$ and $r_1, \ldots, r_k$ be all the vertices of $T$ having $\geq 3$ children. Then computing cone $\mathcal{C}(\sigma)$ can be reduced to computing smaller cones $\mathcal{C}(r_i)$ where each such cone depends only on vertex $r_i$ and its children.

**Example**

$$\varphi(123456) + \varphi(789) + \varphi(10) + \varphi(12345678910) \leq \varphi(123456789) + \varphi(12345610) + \varphi(78910)$$
The cone $\mathbb{L}$

**Theorem**

Let $T$ be the tree corresponding to the cone $\sigma$ and $r_1, \ldots, r_k$ be all the vertices of $T$ having $\geq 3$ children. Then computing cone $\mathbb{L}(\sigma)$ can be reduced to computing smaller cones $\mathbb{L}(r_i)$ where each such cone depends only on vertex $r_i$ and its children.

**Example**

\[ \varphi(7) + \varphi(8) + \varphi(9) + \varphi(789) \leq \varphi(78) + \varphi(79) + \varphi(89) \]
Inductive approach

**Theorem**

If $\mathcal{L} = \mathcal{U}$ for the space of phylogenetic trees with $n - 1$ leaves and $\mathcal{U}$ is contained in the intersection of the cones given by the trees with only one internal vertex and at most $n$ leaves, then $\mathcal{L} = \mathcal{U}$ for the space of phylogenetic trees with $n$ leaves.
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General idea

Theorem (Gibney and Maclagan)

The cones $U$ and $L$ give us a tool to compute an important cone which arises in algebraic geometry.

- A central goal in algebraic geometry is to understand maps $X \to \mathbb{P}^k$, for a projective variety $X$.
- A main tool in studying these maps is the nef cone of $X$.
- Interesting unknown case when $X = \overline{M}_{0,n}$.
Moduli spaces

Definition

The moduli space $M_{0,n}$ is a geometric space whose points correspond to isomorphism classes of smooth curves of genus 0 with $n$ distinct marked points.

- $M_{0,n} = \{\mathbb{P}^1 \text{ with } n \text{ distinct marked points}\} / \text{automorphisms}$.
- Smooth space of dimension $n - 3$.
- Understanding of this space tells us a lot about curves.

Deligne-Mumford compactification $\overline{M}_{0,n}$

Add every curve with $n$ marked points whose group of automorphisms fixing those points is finite.
Divisors (simplified)

**Definition**

A **Divisor** of a variety $X$ is a finite sum of the form $\sum_i a_i D_i$ where each $a_i \in \mathbb{R}$ and each $D_i$ is a codimension 1 subvariety of $X$.

**Definition**

- Let $D$ be a divisor and $C$ a curve in $X$, then $D \cdot C := \sum_i a_i |D_i \cap C|$.
- The **nef cone** of $X$ is the cone generated by divisors such that $D \cdot C \geq 0$ for all curves $C$.

**Example**

$D = \sum_i a_i D_i$ with $a_i \geq 0$ is in the nef cone.
There is a natural embedding of $\overline{M}_{0,n} \hookrightarrow X_{\Delta}$, where $X_{\Delta}$ is the toric variety of the fan of the space of phylogenetic trees.

The cones $\mathcal{L}(\Delta)$ and $\mathcal{U}(\Delta)$ are cones of divisors on $X_{\Delta}$.

Gibney and Maclagan use this embedding to pull back these cones to cones of divisors on $\overline{M}_{0,n}$ which give upper and lower bounds for the nef cone of $\overline{M}_{0,n}$.

If we can prove $\mathcal{L}(\Delta_n) = \mathcal{U}(\Delta_n)$, where $\Delta_n$ is the space of phylogenetic trees with $n$ leaves, then we would have a nice description of the nef cone of $\overline{M}_{0,n}$, which is in general hard to compute.

This technique can also be applied to other projective varieties $X$ for which there is a nice embedding to a toric variety.
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Thank you!

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