A history of the Associahedron

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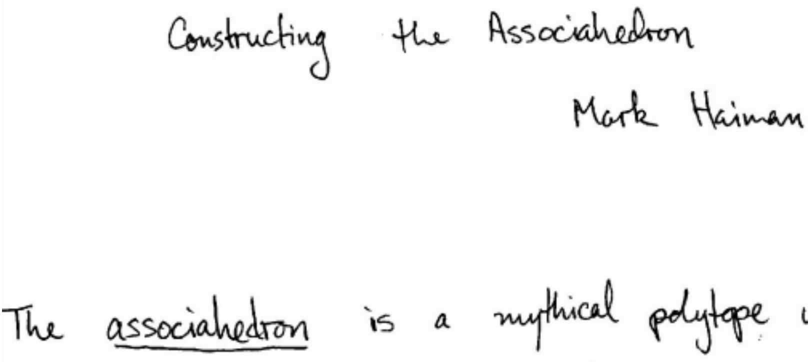


Constructing the Associahedron

Mark Haiman

The associahedron is a mythical polytope

Outline

Catalan numbers

Euler's polygon division problem: In 1751, Euler wrote a letter to Goldbach in which he considered the problem of counting in how many ways can you triangulate a given polygon?

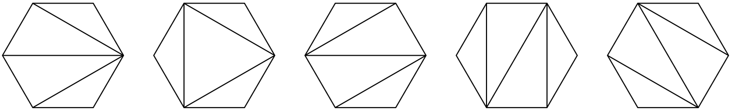


Figure : Some triangulations of the hexagon

Euler gave the following table he computed by hand

n-gon	3	4	5	6	7	8	9	10
C_{n-2}	1	2	5	14	42	152	429	1430

and conjectured that $C_{n-2} = \frac{2 \cdot 6 \cdot 10 \cdots (4n-10)}{1 \cdot 2 \cdot 3 \cdots (n-1)}$

Euler and Goldbach tried to prove that the generating function is

$$A(x) = 1 + 2x + 5x^2 + 14x^3 + 42x^4 + 132x^5 + \ldots = \frac{1 - 2x - \sqrt{1-4x}}{2x^2}$$

Goldbach noticed that $1 + xA(X) = A(x)^{1/2}$ and that this gives an infinite family of equations on its coefficients.

However, they are stuck.

In the late 1750's Euler contacts his frenemy Segner suggesting the triangulation problem, but does not include much details besides the table until the 7-gon.

In 1758 Segner writes a paper with a combinatorial proof of the recurrence

$$C_{n+1} = C_0 C_n + C_1 C_{n-1} + \cdots + C_{n-1} C_1 + C_n C_0$$

He also computes the values of C_n with $n \leq 18$ but makes an arithmetic mistake in C_{13} which causes troubles there on.

Euler uses Segner's recurrence to prove the equation on the generating function.

Euler publishes Segner's paper in the journal of St. Petersburg Academy of Sciences, but with his own summary.

In it he compliments Segner, comments on his numerical mistake and computes C_n for $n \leq 23$.

Hidden in all this is a proof of the product formula, which is equivalent to its more common form $C_n := \frac{1}{n+1}\binom{2n}{n}$. However, they do not publish this proof.

A self-contained proof is not published until 80 years later in France. In 1838 Terquem asks Liouville if he knows a simple way to derive the product formula from Segner's recurrence. Liouville in turn asks this to "various geometers" and it eventually gets solved by Lamé.

Why are they called Catalan numbers?

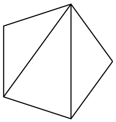
Catalan, a Belgian mathematician, was a student of Liouville. He published a lot of papers about "Segner's numbers" which popularized the subject.

His main contributions to this subject was to observe that $C_n = \binom{2n}{n} - \binom{2n}{n-1}$ and to connect them to counting parenthesized expressions:

Given abcd, the possible parenthesized expressions are $((ab)c)d = (a(bc))d = a((bc)d) = a(b(cd)) = (ab)(cd)$.

The first time they are sort of called Catalan numbers is in a 1938 paper by Bell, in which he calls them "Catalan's numbers" but he doesn't suggest this as a name.
The term becomes standard only after Riodan's book *Combinatorial Identities* was published in 1968.

From triangulations to trees to parentheses



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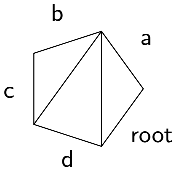

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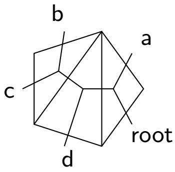
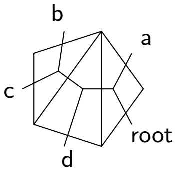
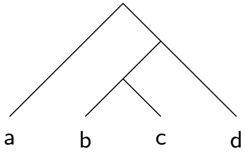


From triangulations to trees to parentheses





a((bc)d)

What do Catalan numbers count?

As of 2013, Stanley had 207 combinatorial interpretations for the Catalan numbers in Stanley's book Enumerative Combinatorics and addendum.

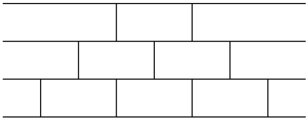
- Full binary trees with $n+1$ leaves
- Standard Young tableaux whose diagram is a $2\times$n-rectangle
- Monotonic paths along the edges of a grid with $n \times n$ square cells, which do not pass above the diagonal.
- http://www-math.mit.edu/~rstan/ec/catadd.pdf

It was later discovered that during the early 1700's a Mongolian mathematician named Minggatu used Catalan numbers to write the series expansion of $\sin(2\theta)$ in terms of $\sin(\theta)$, however there is no evidence he proved that these numbers were integers.

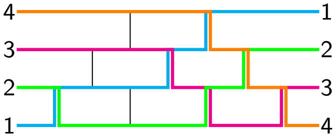
Further reading on Catalan numbers:
http://www.math.ucla.edu/~pak/lectures/Cat/pakcat.htm

Pseudoline Arrangements

Consider a diagram with bricks so that the bounded bricks are ordered in a triangular shape.



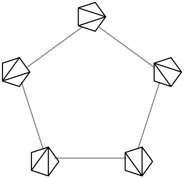
A *pseudoline arrangement* on this diagram is a collection of n pseudolines such that each two have exactly one crossing and no other intersection.



The number of pseudoline arrangements on a brick diagram of this type with n horizontal lines is C_n.

The associahedron

The triangulations of a pentagon can be arranged into a polyhedral complex: two triangulations are adjacent if they differ by one diagonal.



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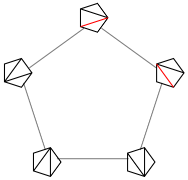

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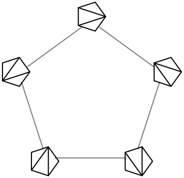


The first person to consider this was Dov Tamari.

He was born in Germany in 1911 as Bernhard Teitler. In 1942 he officially changed his name to Dov Tamari, after spending time in a Jerusalem prison for being a member of a militant underground organization and for being caught with explosives in his room.

Tamari's thesis

Tamari completed his thesis at the Sorbonne in 1951. In this work he considered the set of bracketed sequences as a partially ordered set, and he also thought of this poset as the skeleton of a convex polytope. This partially ordered set is now called the Tamari lattice.

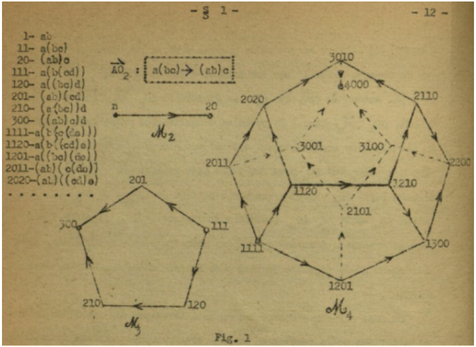


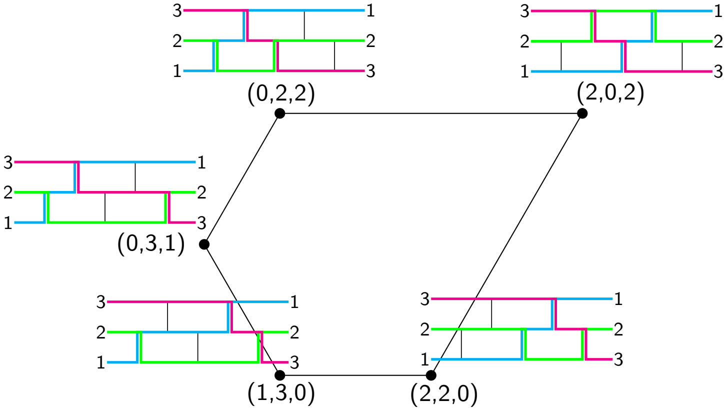
Figure : Picture from Tamari's thesis

Polytopality of the associahedron

Given a finite collection of points in $\mathbb{R}^k$, the *polytope* associated to this set is the minimal convex set containing these points.

A *polytopal realization* of the associahedron K_n is an $(n-2)$-dimensional polytope such that its one skeleton is the Tamari lattice. In other words, the polytope has as many vertices as triangulations of the $(n+1)$-gon and two vertices are adjacent if they differ by one diagonal.

A polytopal realization of K_4



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Still the question remained if one could realize the associahedron as a polytope. Given an n-gon is there an $(n-3)$-dimensional polytope encoding the triangulations of the polygon?

History of the polytopality of the associahedron

- 1951: Tamari defines the associahedron combinatorially.
- 1963: Stasheff realizes the associahedron as a cell complex.
- 1960: Milnor brings a 3D model of the associahedron to Stasheff's thesis.
- Kalai named it the associahedron and posed the problem to Haiman.
- 1984: Haiman gives a construction by inequalities, but it is unpublished and only his hand written notes are available.
- 1989: First construction in print is done by Lee. One of his main interests was to construct a polytope that reflects the symmetries of the n-gon.

Towards geometry

Actually: many realizations have been discovered since. Many mathematicians have been involved in the realizations of the associahedron and its generalizations: Gelfand, Kapranov, Zelevinsky, Postnikov, Loday, Shnider, Sternberg, Fomin, Hohlweg, Thomas, Lange and many, many more.

- Do you care about respecting the symmetry of the *n*-gon and want to see it reflected in the polytope?
- or, do you prefer small integer coordinates for the vertices?
- or, do you care about geometry?

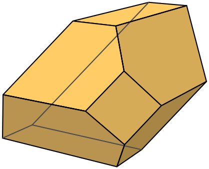


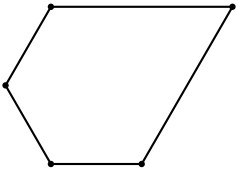
Figure : Loday's 3D associaheron

Why geometry?

A large number of realizations of the associahedron satisfy the following properties:

- the polytope is *rational*, i.e., all edge directions are vectors in $\mathbb{Z}^n$,
- it is *simple*, i.e., there are n edges meeting at each vertex, and
- it is *smooth*, i.e., at each vertex the edge directions are a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

However, to realize the associahedron as a rational polytope one must break the symmetries of the n-gon.



Why geometry?

A large number of realizations of the associahedron satisfy the following properties:

- the polytope is *rational*, i.e., all edge directions are vectors in $\mathbb{Z}^k$,
- it is *simple*, i.e., there are k edges meeting at each vertex, and
- it is *smooth*, i.e., at each vertex the edge directions are a $\mathbb{Z}$-basis of $\mathbb{Z}^k$.

These properties tell us there is an underlying smooth projective toric variety. We will get to this at the end of the talk.

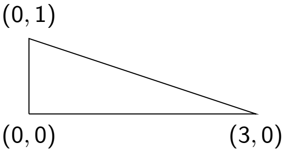


Figure : A non smooth polytope

Outline

The toric variety of the associahedron

Since the associahedron is a nice polytope, “anyone” can construct the toric variety of the associahedron by describing the affine variety around each vertex and then glueing them together.

However, there is a nicer way.

Toric varieties

Definition

A *toric variety* is an algebraic variety X with an algebraic torus $T = (\mathbb{C}^*)^n$ as an open dense subset such that the action of T on T extends to the whole variety.

So to define a toric variety I need to specify

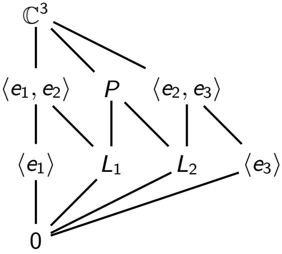
1. the algebraic variety X,
2. the torus $T \subset X$, and
3. an action of T on T.

Given a Delzant polytope, one can construct a smooth projective toric variety associated to this polytope.

The toric variety of the associahedron

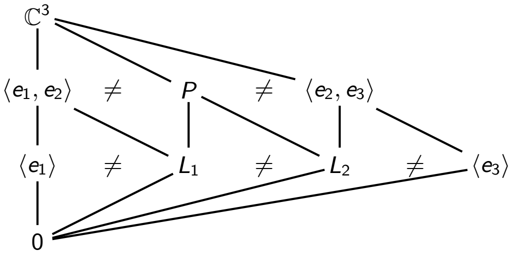
The toric variety of a particular polytopal realization of the 2-dimensional associahedron (a pentagon) is

$$X = \{(L_1, P, L_2) : \text{ the diagram below holds}\}$$



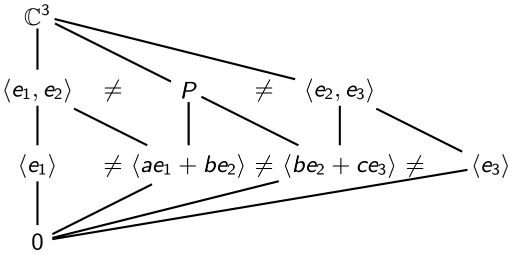
The torus

$$T = \{(L_1, P, L_2) : \text{ the diagram below holds}\}$$



The torus

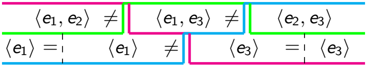
$$T = \{(a, b, c) \in (\mathbb{C}^*)^3 : \text{ the diagram below holds}\}$$

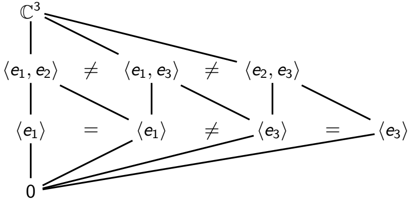


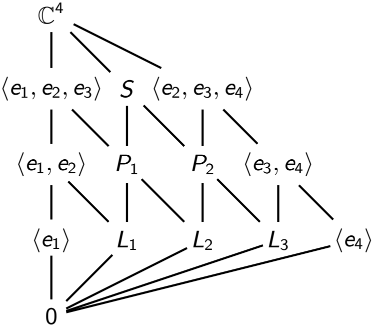
T acts on T by coordinate-wise multiplication

Motivation of relation with associahedron

There is a one-to-one correspondence between pseudoline arrangements and T-fixed points, a.k.a. elements of the toric variety such that each subspace is generated has as basis a subset of $\{e_1, e_2, e_3\}$.







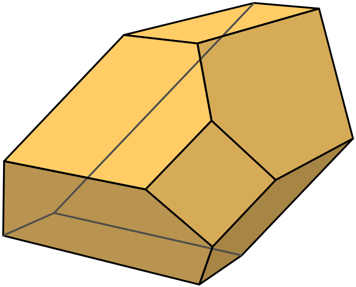
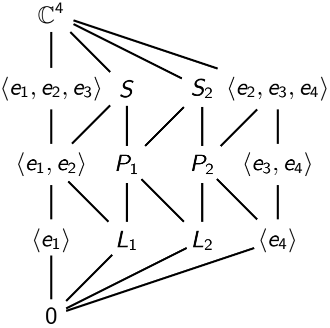


Figure : Loday associahedron



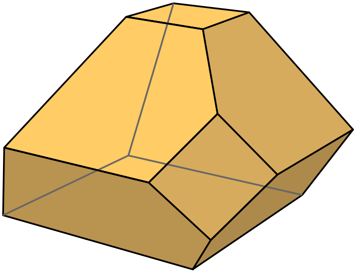


Figure : Chapoton-Fomin-Zelevinsky associahedron

Outline

The associahedron and mathematics

The associahedron is of interest to discrete geometers, for example they want to understand the realization space? (What points in $\mathbb{R}^{n-2}$ have convex hull the associahedron K_n)

For type A cluster algebras, the associahedron encodes mutation between cluster variables.

The associahedron arises in Hubbard and Masur's 1979 paper "Quadratic differentials and foliation's". Here they have a nice proof that K_n is homeomorphic to S^{n-2}.

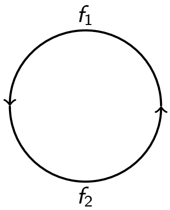
Stasheff and the associahedron

In the 60's, Stasheff invented A_∞-spaces and A_∞-algebras as a tool to study "group-like" spaces.

Let $(X, \star)$ be a topological space with base point $\star$ and let ΩX denote the space of loops in X, i.e. a point in ΩX is a continuous map $f : S^1 \to X$ taking the base point of the circle to $\star$.

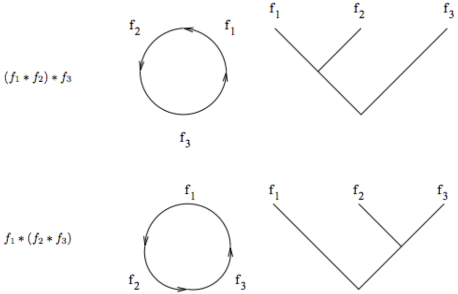
We then have a map $m_2 : \Omega X \times \Omega X \to \Omega X$ defined by $m_2(f_1, f_2) = f_1 * f_2$, i.e. concatenate the two paths.

$f_1 * f_2$



Stasheff and the associahedron

Concatenating is not associative:

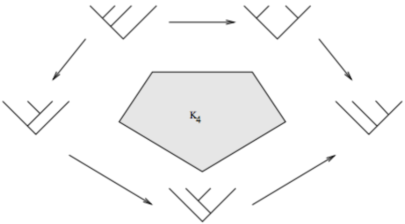


However they are homotopic by a map $m_3 : [0,1] \times \Omega X \times \Omega X \to \Omega X$

Stasheff and the associahedron

When we want to concatenate 4 factors there are 5 possibilities.

Using m_3 we obtain two paths of homotopies joining $((f_1 * f_2) * f_3) * f_4$ with $f_1 * (f_2 * (f_3 * f_4))$



These paths are homotopic by the map $m_4 : K_4 \times (\Omega X)^4 \to \Omega X$, where K_4 denotes the pentagon bounded by these two paths.

Stasheff and the associahedron

To compose 5 factors we get 14 possibilities corresponding to the 14 binary trees with 5 leaves. Using m_3 and m_4 we obtain paths using the compositions and faces linking the paths. The resulting sphere is the boundary of K_5.

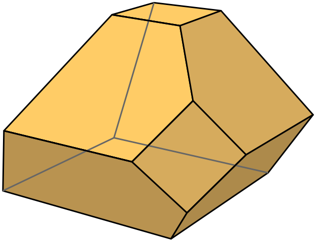


Figure : The associahedron K_5

A_∞-spaces

Stasheff defined cell complexes K_n of dimension $n-2$ for all $n \geq 2$ and defined an *A_∞-space* to be a topological space Y endowed with maps $m_n : K_n \times Y^n \to Y$ for $n \geq 2$ satisfying some suitable compatibility conditions and admitting a "unit".

A topological space that admits the structure of an A_∞-space and whose connected components form a group is homotopy equivalent to a loop space.

Read more about this in Keller's introductory notes: `http://arxiv.org/pdf/math/9910179v2.pdf` (I took some pictures from this paper for this talk.)

Tesselations by associahedra

What happens if we admit commutation among the variables we are multiplying?
$(ab)c = (ba)c \mapsto b(ac)$

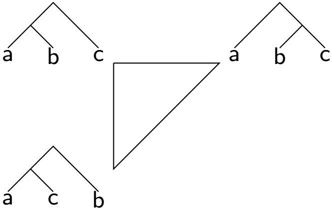
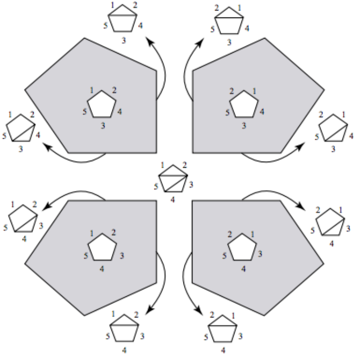
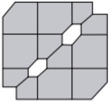


Figure : Get all binary trees with their leaves labelled with three K_3.

That means we can change the labels on the sides of the n-gon, or consider all binary trees with their leaves labeled

Nice pictures by Satyan Devadoss





Space of phylogenetic trees

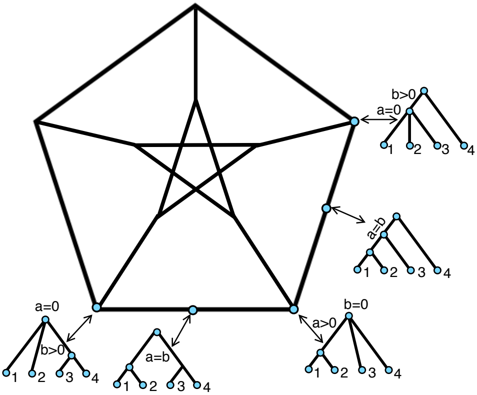
A *rooted tree* is a graph that has no cycles and which has a vertex of degree at least 2 labelled as the root of the tree.

Its *leaves* are all the vertices of degree 1. We label them from 1 to n.

A *phylogenetic tree* is a rooted tree with no vertices of degree 2 such that each internal edge is assigned a nonnegative length.

The space $\mathcal{T}_n$ *of phylogenetic trees* on n leaves parametrizes all such trees with nonnegative internal edge measures.

Space of phylogenetic trees $\mathcal{T}_4$



Space of phylogenetic trees

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The space *$\mathcal{T}_n$ of phylogenetic trees* on n leaves parametrizes all such trees with nonnegative internal edge measures.

The space of phylogenetic trees is used to encode evolutionary relationships between species. Each leaf is a species and the tree structure records their evolutionary history.

The space of phylogenetic trees $\mathcal{T}_n$ is “tiled” by $n!/2$ associahedra.

Tesselations by associahedra in other areas

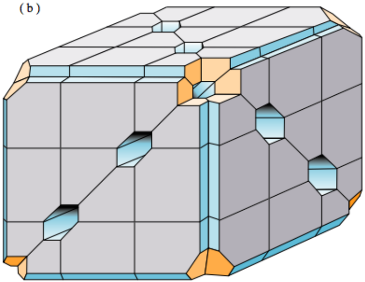

Figure : The moduli space $\overline{M}_{0,n}(\mathbb{R})$ is tiled by 60 associahedra

Thank you!