Bott-Samelson varieties and combinatorics

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Geometric and topological combinatorics:
Modern techniques and methods

MSRI

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Based on:
arXiv:1404.467
arXiv:1605.05613 (with O. Pechenik, B. Tenner, and A. Yong)
arXiv:1708.06663 (with B. Wyser and A. Yong)

Schur-Horn Theorem

$\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$.

$\mathcal{O}_\lambda := \{$Hermitian matrices with eigenvalues $(\lambda_1, \ldots, \lambda_n)\}$.

Schur-Horn Theorem. There is a matrix in $\mathcal{O}_\lambda$ with diagonal entries $(d_1, \ldots, d_n)$ if and only if $(d_1, \ldots, d_n) \in \mathcal{P}_\lambda$.

$\mathcal{P}_\lambda := \mathrm{conv}\{(\lambda_{w_1}, \ldots, \lambda_{w_n}) \mid w \text{ a permutation of } [n]\}$.

Atiyah-Guillemin-Sternberg Convexity Theorem

Atiyah-Guillemin-Sternberg Convexity Theorem. Suppose that M is a compact connected symplectic manifold with an action of a torus T and moment map $\Phi : M \longrightarrow \mathfrak{t}^*$ for this action.
Then $\Phi(M) = \mathrm{conv}\{\Phi(p) \mid p \text{ is a T-fixed point of M}\}$.

Schur-Horn Theorem. There is a matrix in $\mathcal{O}_\lambda$ with diagonal entries $(d_1, \ldots, d_n)$ if and only if $(d_1, \ldots, d_n) \in \mathcal{P}_\lambda$.

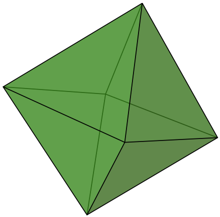
$\mathrm{T} = \{\text{diagonal matrices}\}$ acts on $\mathcal{O}_\lambda$ by conjugation.

The T-fixed points are diagonal matrices.

The map $\Phi : \mathcal{O}_\lambda \longrightarrow \mathbb{R}^n$ defined by $\Phi(H) = (H_{11}, \ldots, H_{nn})$ is a moment map and $\Phi(\mathcal{O}_\lambda) = \mathcal{P}_\lambda$.

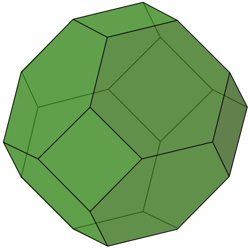
Other moment polytopes

The hypersimplex is the moment polytope for the Grassmannian.



The matroid polytope of V's matroid is the moment polytope for the T-orbit closure of V in the Grassmannian.

The permutahedron is the moment polytope for the flag manifold.

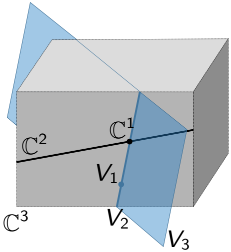


The bruhat interval polytopes of E. Tsukerman and L. Williams are moment polytopes for Schubert and Richardson varieties in the flag manifold.

Schubert varieties

The **flag manifold** is
$\text{Flag}_n = \{(V_1, \ldots, V_n) : V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n\}$ where each V_i is an i-dimensional vector subspace of $\mathbb{C}^n$.

A **Schubert variety** consists of the $(V_1, \ldots, V_n) \in \text{Flag}_n$ satisfying some bounds on the dimension of $V_i \cap \mathbb{C}^j$ for all i, j.



Most Schubert varieties are not smooth.

Singularities of Schubert varieties

The **Kazhdan-Lusztig polynomial** $P_{v,w}(q)$ measures how bad the singularity of the Schubert variety X_w is at the point e_v.

X_w is (rationally) smooth at e_v if and only if $P_{v,w}(q) = 1$.

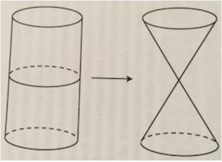
The Kazhdan-Lusztig polynomials are the Poincaré polynomials for intersection homology of Schubert varieties.

All the coefficients are positive. There is no combinatorial proof of positivity.

Resolutions of singularities

Resolutions of singularities provide tools to compute Kazhdan-Lusztig polynomials.

A **resolution** of X is a polynomially defined surjective map $\pi : Y \to X$ such that Y is smooth and π is invertible at all smooth points on X.



Since Y is smooth its intersection homology is simply its homology.

When π is a small map, the intersection homology of Y is isomorphic (as a group) to the intersection homology of X.

Problem (A. Zelevinsky '83)**.** Describe the Schubert varieties that admit small resolutions.

Bott-Samelson resolutions

H.C. Hansen '73 and M. Demazure '74 independently presented the Bott-Samelson resolutions of Schubert varieties.

A. Zelevinsky '83 gave a resolution for Schubert varieties in the Grassmannian, presented as a configuration space of vector spaces prescribed by dimension and containment conditions.

P. Magyar '98 gave a new description of the Bott-Samelson resolution in the same spirit.

In general, these resolutions are not small.

Preview of Main Results

Theorem (E.,Pechenik,Tenner,Yong)**.** The Bott-Samelson resolution of singularities of a Schubert variety consists of vector spaces arranged on a rhombic tiling of the Elnitsky polygon.

Theorem (E.)**.** The toric variety of an associahedron of C. Hohlweg, C. Lange, and H. Thomas equals the general fiber of certain Bott-Samelson map.

Theorem (E.-Wyser-Yong)**.** The Barbasch-Evens resolution of singularities of a symmetric orbit closure is a configuration space of vector spaces prescribed by dimension and containment conditions.

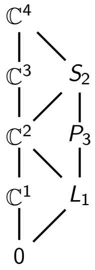
Theorem (E.-Wyser-Yong)**.** The moment polytope of the Barbasch-Evens resolution is the convex hull of certain reflections of the moment polytope of the Bott-Samelson resolution.

Magyar's construction of the Bott-Samelson manifold

$I = (i_1, i_2, \ldots, i_k)$ where $i_j \in [n]$.

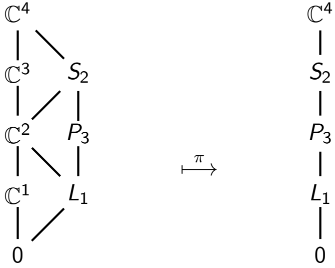
The **Bott-Samelson manifold** is $BS_I \subset \prod_{j=1}^{k} Gr(d_{i_j}, n)$.

$BS_{(1,3,2)} = \{(L_1, S_2, P_3) : \text{the following incidences hold}\}$
$\subset Gr(1,4) \times Gr(3,4) \times Gr(2,4)$.



Magyar's construction of the Bott-Samelson resolution

The Bott-Samelson map $\pi : BS_{(1,3,2)} \to \mathrm{Flag}_4$ is

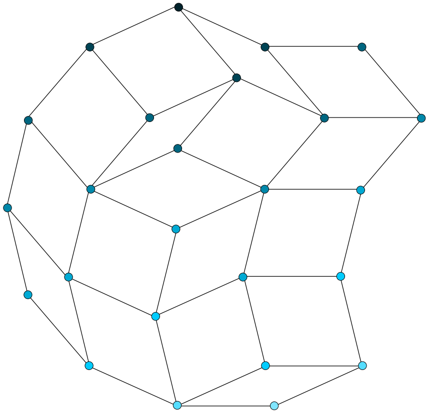


The image of π is a Schubert variety X_w.

If $w = s_{i_1} \cdots s_{i_k}$ is reduced, where s_i transposes i and $i+1$ then $\pi : BS_{(i_1,\ldots,i_k)} \to X_w$ is a resolution of singularities .

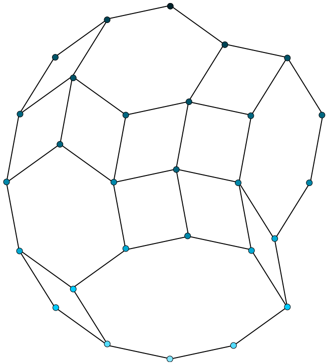
Bott-Samelson resolutions and tilings

Theorem (E.,Pechenik,Tenner,Yong)**.** If $\pi : BS_I \to X_w$ is a resolution of singularities, then BS_I consists of vector spaces arranged on a rhombic tiling of the Elnitsky polygon of a permutation w.



Other resolutions

Theorem (E., Pechenik, Tenner, Yong)**.** Given a zonotopal tiling ζ of the Elnitsky polygon of w, its corresponding generalized Bott-Samelson manifold BS_ζ together with the map $\pi_\zeta : BS_\zeta \to X_w$ is a resolution of singularities.



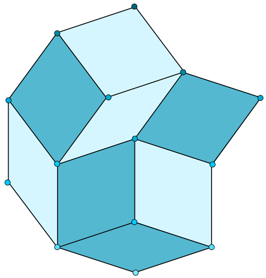
Torus action

The torus $\mathsf{T} = (\mathbb{C}^*)^n$ acts on $\mathbb{C}^n$ by component-wise multiplication.

T acts on $Gr(d, n)$ by acting on the elements of a basis.

T acts on BS_I diagonally.

Proposition (E.,Pechenik,Tenner,Yong)**.** The T-fixed points of BS_I are in bijection with bipartitions of the rhombi.



Symplectic structure

$Gr(d, n)$ is a symplectic manifold and has moment map.

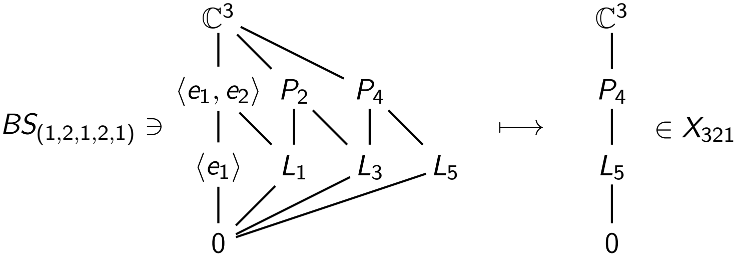
$BS_{(i_1,\dots,i_k)}$ inherits a symplectic structure and moment map from $\prod_{j=1}^{k} Gr(d_{i_j}, n)$.

By the Atiyah-Guillemin-Sternberg convexity theorem $\Phi(BS_I)$ is the convex hull of the images of the T-fixed points.

Brick manifolds

Let p_w be the only T-fixed general point of X_w.

The **brick manifold** $\mathcal{B}_I$ is the fiber $\pi^{-1}(p_w)$ of the Bott-Samelson map $\pi : BS_I \to X_w$.



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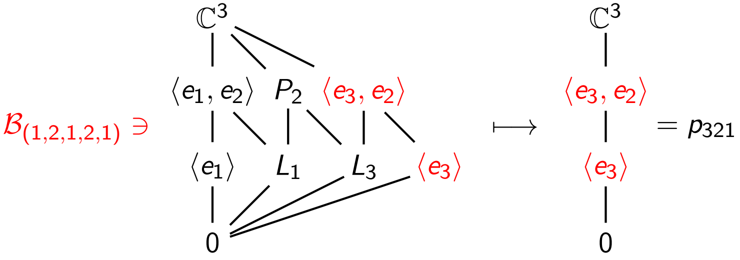


The toric variety of the associahedron

$\mathcal{B}_I$ inherits a symplectic structure and moment map from BS_I.

Theorem (E.)**.** The moment polytope of the brick manifold is the brick polytope of V. Pilaud, F. Santos and C. Stump.

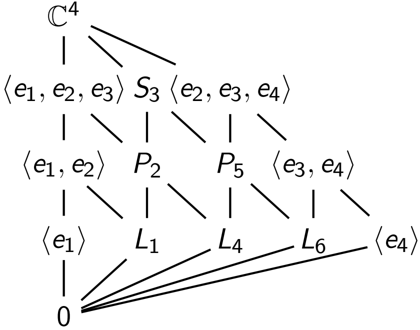
Every associahedron of C. Hohlweg, C. Lange, and H. Thomas is the moment polytope of $\mathcal{B}_I$ for certain I.

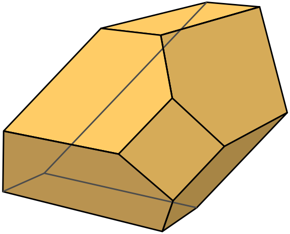
For these I, $\dim(\mathcal{B}_I) = \dim(\mathrm{T})$. Therefore:

Theorem (E.)**.** The toric variety of an associahedron of C. Hohlweg, C. Lange, and H. Thomas equals $\mathcal{B}_I$ for certain I.

Loday associahedron

The brick manifold $\mathcal{B}_{(1,2,3,1,2,3,1,2,1)}$ is the toric variety of Loday's 3D associahedron.





K-orbit closures

Let θ be an involution of $GL_n(\mathbb{C})$, and let K be the subgroup of fixed points of the involution.

K acts on Flag_n with finitely many orbits.

Most K-orbit closures are not smooth.

For $\text{K} = GL_p \times GL_q$ a K-orbit closure consists of $(V_1, \ldots, V_n) \in \text{Flag}_n$ satisfying some bounds on the dimensions of $V_i \cap \mathbb{C}^j$ and $V_i \cap (\mathbb{C}^j)^\perp$ for all i, j.

Singularities of K-orbit closures

The **Kazhdan-Lusztig-Vogan polynomials** are a family of polynomials associated to a symmetric pair (G, K).

They measure how bad the singularity of a K-orbit closure is at a point.

They are the Poincaré polynomials for intersection homology of K-orbit closures.

All the coefficients are positive. There is no combinatorial proof of positivity.

Resolutions of singularities provide tools to compute Kazhdan-Lusztig-Vogan polynomials.

Resolutions of singularities for K-orbit closures

D. Barbasch and S. Evens '94 presented resolutions for K-orbit closures analogous to the Bott-Samelson resolutions.

Theorem (E.-Wyser-Yong)**.** A Barbasch-Evens variety for a symmetric pair (G, K) is isomorphic as a K-variety to a configuration space of vector spaces prescribed by dimension and containment conditions.

We call this construction a **Barbasch-Evens-Magyar variety**.

Barbasch-Evens-Magyar varieties for $K = GL_p \times GL_q$

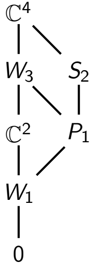
Fix Y_0 a closed K-orbit and $I = (i_1, \ldots, i_k)$ where $i_j \in [n]$.

The Barbasch-Evens-Magyar variety for Y_0 and I is

$$BEM_{Y_0,I} \subset \prod_{j=1}^{n-1} Gr(j,n) \times \prod_{j=1}^{k} Gr(d_{i_j}, n).$$

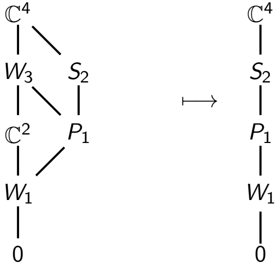
For $Y_0 = \{(W_1, W_2, W_3) \in \mathrm{Flag}_4 \mid W_2 = \mathbb{C}^2\}$ and $I = (2,3)$,

$BEM^{Y_0,(2,3)} =$



Barbasch-Evens-Magyar varieties for $K = GL_p \times GL_q$

The map is $\pi : BEM_{Y_0,(2,3)} \to \mathrm{Flag}_4$ is



The image of π is a K-orbit closure Y.

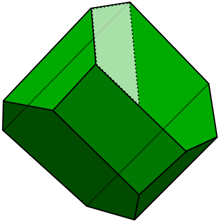
If $w = s_{i_1} \cdots s_{i_k}$ is reduced, $\dim(Y) = \dim(BEM_{Y_0,I})$, and Y is multiplicity-free then π is a resolution of singularities.

Moment polytopes

$BEM_{Y_0,I}$ inherits a symplectic structure and moment map from $\prod_{j=1}^{n-1} Gr(j,n) \times \prod_{j=1}^{k} Gr(d_{i_j}, n)$.

Theorem (E.-Wyser-Yong)**.** The moment polytope of $BEM_{Y_0,I}$ is the convex hull of certain S_n-reflections of the moment polytope of BS_I.

The moment polytope of $BEM_{Y_0,(2,3)}$ is the convex hull of four reflections in $\mathbb{R}^3$ of the moment polytope of the Bott-Samelson variety $BS_{(2,3)}$ (white).



Summary

The role of the permutahedron in the Schur-Horn theorem can be explained in terms of Hamiltonian symplectic manifolds and their moment maps.

Schubert varieties, and their analogues, the K-orbit closures are interesting singular varieties.

They have combinatorially described resolutions of singularities.

These descriptions allow one to deepen our understanding of the singular structure of Schubert varieties and K-orbit closures, and to study their moment polytopes.

Thank you!