Non compact symplectic toric manifolds.

This is joint work with Yaël Karshon.

Consider a manifold $M$ with a symplectic form $\omega$. Then any function (Hamiltonian) $\mu: M \to \mathbb{R}$ defines a vector field, hence a flow $\{\varphi_t\}$.

I'll call $\mu \in C^0(M)$ periodic if $\varphi_{t+1} = \varphi_t \cdot t$, i.e. $\{\varphi_t\}$ is periodic of period 1.

Then $\{\varphi_t\}$ defines an action of $S^1 \simeq \mathbb{R}/\mathbb{Z}$ on $M$:

$$[t] \cdot m := \varphi_t(m) \quad \forall [t] \in \mathbb{R}/\mathbb{Z}.$$ 

Next suppose we have $n = \frac{1}{2} \dim M$ periodic Hamiltonians $\lambda_i : M \to \mathbb{R}$ with $\{\lambda_i\} = 0$ (Poisson bracket).

Then the flows $\varphi_i$ and $\varphi^\mu$ commute and

$$\mu = \lambda_1 \ldots \lambda_n : M \to \mathbb{R}^n \quad \text{(moment map)}$$

defines an action of $(\mathbb{R}/\mathbb{Z})^n \cong \mathbb{R}^n/\mathbb{Z}^n = T^n$.

$$[t] \cdot [t] \cdot [t] \cdot m = (\varphi_{t(t)} \ldots \varphi_{t(t)}) \cdot m.$$

We call this action of $T^n$ on $(M, \omega, \mu)$ Hamiltonian.

The triple $(M, \omega, \mu: M \to \mathbb{R}^n)$ is called a symplectic $T^n$-toric manifold if $n = \frac{1}{2} \dim M$ and $\mu$ vanishes on an open dense set (i.e., action of $T^n$ is effective).

Facts:
- Connected components of fibers of $\mu$ are $T^n$-orbits.
- The orbits of $T^n$ are isotropic.
- $\dim$ count $\Rightarrow$ generic fibers are Lagrangian.

Moment maps for Hamiltonian torus actions on compact connected manifolds have remarkable properties.
Thm (Atiyah, Guillemin-Sternberg)

\((\mathcal{M}, \omega, \mu : \mathcal{M} \to \mathbb{R}^n)\) Hamiltonian \(T^n\) action on a compact conn. symplectic manifold. Then
1) \(\mu(M)\) is a polytope [convexity]
2) fibers of \(\mu\) are connected. [connectedness]

The case of symplectic toric manifolds is even more rigid.

Thm (Delzant) let \((\mathcal{M}, \omega, \mu : \mathcal{M} \to \mathbb{R}^n)\) be a compact connected symplectic \(T^n\)-toric manifold. Then
1) \(\mu(M)\) is a unimodular ("Delzant") polytope
2) Unimodular polytopes in \(\mathbb{R}^n\) classify equivalence classes of (compact connected) symplectic \(T^n\)-toric manifolds.

\(\Delta \subset \mathbb{R}^n\) is unimodular if a vertex \(p\) of \(\Delta\) is a basis \(\{v_1, \ldots, v_i\}\) of \(\mathbb{R}^n\) so that edges of \(\Delta\) coming out of \(p\) lie on rays \(p + tv_i\), with \(t \geq 0\).

Note that Delzant's theorem has two parts:

- **Existence**: Given a unimodular polytope \(\Delta \subset \mathbb{R}^n\) [compact and connected], symplectic \(T^n\)-toric manifold \((\mathcal{M}, \omega, \mu : \mathcal{M} \to \mathbb{R}^n)\) with \(\mu(M) = \Delta\)

- **Uniqueness**: if \((\mathcal{M}, \omega, \mu_1), \omega = 1, 2\) are two symplectic \(T^n\)-toric compact connected manifolds with \(\mu_1(M_1) = \mu_2(M_2)\)

Then if \(T^n\)-equivariant symplecto \(\phi : \mathcal{M}_1 \to \mathcal{M}_2\) with \(M_2 \circ \phi = \mathcal{M}_1\),
Note also: At the heart of the proof of Aiyah, Guillemin-Sternberg Thom's Bott-Morse theory is if \( h: M \to \mathbb{R} \) is a periodic Hamiltonian, then \( h \) is Bott-Morse with all indices even.

Now suppose \((M, w, \mu: M \to \mathbb{R}^n)\) is a symplectic \( T^n \)-tropic manifold connected but not compact. Things look hopeless:
- no Morse theory
- fibers of \( \mu \) need not be connected
- \( \mu(M) \) tells you little.

However, if we change point of view things look better. Facts: \( M/T^n \) is a manifold with corners
- \( \mu: M \to \mathbb{R}^n \) descends to \( \overline{\mu}: M/T^n \to \mathbb{R}^n \),
  which is locally unimodular embedding
  maps corners of \( M/T^n \) to unimodular cones.

**Thm (Kurchon-L)** Let \((M, w, \mu: M \to \mathbb{R}^n)\) be a (connected) symplectic \( T^n \)-tropic manifold. Then

"Existence." Given a unimodular local embedding of a manifold with corners \( \psi: W \to \mathbb{R}^n \), there exist a symplectic \( T^n \)-tropic manifold \((M, w, \mu)\) with \( M/T^n = W \) and \( \overline{\mu} = \psi \).

"Uniqueness." The equivalence classes of symplectic \( T^n \)-tropic manifolds \((M, w, \mu)\) with \( M/T^n = W \) and \( \overline{\mu} = \psi \) are in bijective correspondence with the elements of \( H^2(W, \mathbb{Z}^n \times \mathbb{R}) \).
Comments

"Uniqueness" is not hard with existing technology.
"Existence" is harder, in contrast to the compact case (Delzant's thm).
Delzant constructs toric manifolds from a polytope $\Delta$ as symplectic quotients. In the non-compact case there is no polytope.

So we needed a new idea.
Fix a unimodular local embedding $\psi: W \to \mathbb{R}^n$ of an $n$-dim manifold with corners.
We have a category $\text{STM} (W \to \mathbb{R}^n)$ of symplectic $\mathbb{T}^n$-toric manifolds with orbit space $W$ and orbital moment map $\Psi$.
We consider a new category $\text{STB} (W \to \mathbb{R}^n)$ of principal $\mathbb{T}^n$-bundles (with corners) over $W$ with symplectic forms and orbital moment map $\Psi$.

$\exists W = \mathbb{R}^n \xrightarrow{\psi} \mathbb{R}^n$

$\text{STM} \xrightarrow{\text{STB}}$

$\mathbb{R}^n \xrightarrow{t \to \theta} \mathbb{R}^n$

$W = d(t + \theta)$

Theorem (Kashiwara). There is a functor $\text{STB} \to \text{STM}$, which is an equivalence of categories. Since $\text{STB} \neq \emptyset$, $\text{STM} \neq \emptyset$ as well.

In fact more is true: \text{STB is a monoidal category} and \text{STM is an STB-torsor}.

Note: This explains the bijection $\text{iso classes of objects in } \text{STM} \cong H^2 (W, \mathbb{Z}^n \ltimes \mathbb{R})$.

We have a bijection: \text{iso classes of objects in } \text{STB} \cong H^2 (W, \mathbb{Z}^n \ltimes \mathbb{R})$.