Next, we’d like to relate compact-open topology to the topology of uniform convergence on compact sets.

Let $X$ be a compact space, $(Y, d)$ a metric space. Then for all $f, g \in C^0(X, Y)$ the function $x \mapsto d(f(x), g(x))$ is continuous (since $d : Y \times Y \to [0, \infty)$ and $(f, g) : X \to Y \times Y$ are continuous).

Since $X$ is compact, the set $\{d(f(x), g(x)) \mid x \in X\}$ is compact. (We denote $d_\infty (f, g) := \sup \{d(f(x), g(x)) \mid x \in X\}$.)

**Lemma 29.1** $d_0 : C^0(X, Y) \times C^0(X, Y) \to [0, \infty)$, $d_\infty (f, g) = \sup \{d(f(x), g(x)) \mid x \in X\}$ is a metric.

Proof (sketch) we check the triangle inequality: $\forall f, g, h \in C^0(X, Y), \forall x \in X$

\[ d(f(x), g(x)) \leq d(f(x), h(x)) + d(h(x), g(x)) \leq d_\infty (f, h) + d_\infty (h, g). \]

$\therefore d_\infty (f, g) = \sup \{d(f(x), g(x)) \mid x \in X\} \leq d_\infty (f, h) + d_\infty (h, g).$

The other two checks are even easier.

**Definition** Let $X$ be a LCH space, $(Y, d)$ a metric space. A net $\{f_\alpha \}_{\alpha \in A}$ in $C^0(X, Y)$ converges to $f \in C^0(X, Y)$ uniformly on compact sets if $f_\alpha |_K \to f |_K$ in $C^0(K, Y)$ w.r.t. $d^K_\infty : C^0(K, Y) \times C^0(K, Y) \to [0, \infty)$ for compact $K \subseteq X$.

$$d^K_\infty (f, g) := \sup \{d(f(x), g(x)) \mid x \in K \land f, g \in C^0(K, Y)\}$$

We’ll show $f_\alpha \to f$ uniformly on compact sets $\iff f_\alpha \to f$ in c.-o. topology on $C^0(X, Y)$.

**Exercise** Let $(Y, d)$ be a metric space, $C \subseteq Y$ compact, $U \subseteq Y$ open with $C \subseteq U$. Then $\exists \varepsilon > 0$ so that $B_\varepsilon (c) := \{y \in Y \mid d(y, c) < \varepsilon\}$ for some $c \in Y$ is contained in $U$.

Mimic proof of Lebesgue Lemma.

**Lemma 29.2** Let $X$ be a compact Hausdorff space, $(Y, d)$ a metric space. Then $(C^0(X, Y), d_\infty)$ is homeomorphic to $(C^0(X, Y), T_{\text{compact-open}})$. 
Proof. Fix \( f \in C^0(X,Y) \) and consider an open ball \( B\varepsilon(f) \subset C^0(X,Y) \); 
\[
B\varepsilon(f) = \{ g \in C^0(X,Y) | d_\infty(f, g) < \varepsilon \}.
\]
For any \( x \in X \) there is a compact nbhd \( N_x \) with \( N_x \subset f^{-1}(B\varepsilon(f(x))) \).

Since \( X \) is compact, \( \exists k \in \mathbb{N} \) such that \( x_1, \ldots, x_k \in X \) with \( X = N_{x_1} \cup \cdots \cup N_{x_k} \).

Let \( V = \bigcap_{i=1}^{k} M(N_{x_i}, B\varepsilon(f(x_i))) < C^0(X,Y) \); \( V \) is a nbhd of \( f \) in compact-open topology. Suppose \( g \in V \). Then \( \forall x \in N_{x_i}, \ g(x) \in B\varepsilon(f(x_i)), \) i.e. 
\[
x \in N_{x_i} \Rightarrow d(g(x), f(x_i)) < \varepsilon/3
\]
\[
\Rightarrow \forall x \in N_{x_i}, \quad d(g(x), f(x_i)) < \varepsilon/3 + \varepsilon/3 = \varepsilon/2.
\]
Since \( \bigcup_{i=1}^{k} N_{x_i} = X \), 
\[
\Rightarrow d_L(f, g) = \sup \{ d(f(x), g(x)) | x \in X \} \leq \frac{\varepsilon}{2} < \varepsilon \Rightarrow g \in B\varepsilon(f).
\]
\[
\Rightarrow V \subset B\varepsilon(f).
\]
\[
\Rightarrow \text{Any set in } T_{d_\infty} \text{ is in } T_{\text{compact-open}}.
\]

Conversely, suppose \( f \in \bigcap_{i=1}^{k} M(K_i, U_i) \) for some \( K_1, \ldots, K_k \subset X \) compact, \( U_1, \ldots, U_k \subset Y \) open.

Then \( f(K_i) \subset U_i \forall i \Rightarrow \exists \varepsilon_i > 0 \text{ s.t. } B\varepsilon(f(K_i)) \subset U_i \forall i \). Let \( \varepsilon = \min \{ \varepsilon_1, \ldots, \varepsilon_k \} \).

Then \( B\varepsilon(f(K_i)) \subset U_i \forall i \). 

Now if \( g \in B\varepsilon(f) \) then \( d(g(x), f(x)) < \varepsilon \forall x \in X \).

\[
\Rightarrow \forall x \in K_i, \quad g(x) \in B\varepsilon(f(K_i)) \subset U_i \Rightarrow g \in M(K_i, U_i) \forall i.
\]
\[
\Rightarrow B\varepsilon(f) \subset \bigcap_{i=1}^{k} M(K_i, U_i)
\]
\[
\Rightarrow \text{Any set in } T_{\text{compact-open}} \text{ is in } T_{d_\infty}.
\]

Theorem 30.1. Let \( X \) be a LCH space, \( (Y, d) \) a metric space. Let \( \{ f_n \}_{n \in \mathbb{N}} \) be a net in \( C^0(X,Y) \). Then \( f_n \rightharpoonup f \in C^0(X,Y) \) in compact-open (c-o) topology \( \iff \)
\[
\forall K \subset X \text{ compact}, \quad f_n|_K \rightharpoonup f|_K \text{ in the metric space } (C^0(K,Y), d_\infty).
\]
Proof ($\Rightarrow$) Suppose $f_\alpha \to f$ in c.o. topology. By Lemma 28.6, the restriction map $c^0(X,Y) \to c^0(K,Y)$, $h \mapsto h|_K$ is continuous (w.r.t. the c.o. topologies). Continuous maps send convergent nets to convergent nets. $\Rightarrow f|_K \to f|_K$ in c.o. topology on $c^0(K,Y)$.

By Lemma 29.2, the c.o. topology and the metric topologies on $c^0(K,Y)$ agree. Hence $K \subseteq X$ compact $f|_K \to f|_K$ in $(c^0(K,Y), d_\infty)$.

($\Leftarrow$) Suppose $K \subseteq X$ compact, $f|_K \to f|_K$ in the metric topology.

Then $f|_K \to f|_K$ in the c.o. topology.

Let $W$ be a nbhd of $f$ in $c^0(X,Y)$. Then $\exists K \subseteq X$ compact, $U \subseteq Y$ open so that $f \in M(K,U) \subseteq W$. Then $f(K) \subseteq U$.

Recall that $\exists \varepsilon > 0$ so that $B_\varepsilon (f(K)) = \bigcup_{y \in f(K)} B_\varepsilon (y)$ is contained in $U$.

Since $f|_K \to f|_K$ in $(c^0(K,Y), d_\infty)$, $\exists \varepsilon > 0$ so that if $\beta < \alpha$, then $d_\infty (f|_K, f|_K) < \varepsilon$.

$\Rightarrow \forall x \in K$, $d(f(x), f(x)) < \varepsilon$.

$\Rightarrow f(x) \in B_\varepsilon (f(K)) \subseteq B_\varepsilon (f(K)) \subseteq U$ for all $x \in K$ and all $\alpha$ with $\beta < \alpha$.

$\Rightarrow f_\alpha (K) \subseteq U$ for all $\alpha$ with $\beta < \alpha$.

$\Rightarrow$ for all $\alpha$ with $\beta < \alpha$, $f_\alpha \in M(K,U) \subseteq W$.

$\therefore f_\alpha \to f$ in c.o. topology.

Stone-Čech compactification.

Recall that if $X$ is an LCH space then we can give $X^+ = X \cup \{\infty\}$ a topology so that

(1) $X^+ \subseteq X$ is compact and (2) $X \subseteq X^+ \subseteq X^+ \cup \{\infty\}$ is an embedding.

Moreover if $X$ is not compact then $i(X)$ is dense in $X^+$ and therefore $i : X \to X^+$ is a compactification of $X$, a 1-point compactification.

A given space may have many compactifications and $i : X \to X^+$ is the smallest: $X^+$ has only one extra point compare to $X$.

There is also the biggest - the Stone-Čech compactification.
What's the idea? Recall:

**Lemma 17.2** Suppose $X$ is a $T_1$ space (points are closed) and $\{ f_x : X \to [0,1] \}_{x \in A}$ a collection of continuous functions that separates points and closed sets: $\forall x \in X \neq x' \in X$ closed with $x \not\in C$, $\exists x \in A$ st $f_x(x) = 1$, $f_x | C = 0$.

Then $F : X \to [0,1]^A$, $F(x) = (f_x(x))_{x \in A}$ is an embedding ($[0,1]^A$ to given product topology).

Suppose $X$ is completely regular. Then $M := C^0(X, [0,1])$ is a collection of functions on $X$ that separate points and closed sets. Hence by 17.2 the map

$$\eta_x : X \to [0,1]^M \equiv [0,1]^{C^0(X, [0,1])}, \quad \eta_x(x) : C^0(X, [0,1]) \to [0,1], f \mapsto f(x)$$

is an embedding.

On the other hand, $[0,1]^M$ is compact for any set $M$ (and Hausdorff). Hence

$$\beta(x) := \eta_x(x) \in [0,1]^{C^0(X, [0,1])} \text{ is compact.}$$

**Definition** The Stone–Čech compactification of a completely regular space $X$ is the embedding $\eta_x : X \to \beta(x) = \eta_x(x)$. 