The compact-open (c.o.) topology on \( C^0(\mathcal{X}, Y) \) is the topology generated by the sets 

\[
M(K, U) = \{ f : X \to Y \mid f(K) \subseteq U \},
\]

where \( K \subseteq X \) is compact and \( U \subseteq Y \) is open.

**Lemma 28.1** Suppose \( X \) is LCH (loc. compact Hausdorff). Then \( \text{ev} : C^0(\mathcal{X}, Y) \times X \to Y, (f, x) \mapsto f(x) \) is continuous.

**Corollary 28.2** Suppose \( X \) is LCH. Then 

\[
\forall h \in C^0(\mathcal{Z}, C^0(\mathcal{X}, Y)), \; \overline{h} := \text{ev}_0(h, \cdot, x) \in C^0(\mathcal{Z} \times X, Y).
\]

**Remark:** Suppose \( k : \mathcal{X} \times X \to Y \) is continuous. Then \( \forall x \in \mathcal{X}, \; \overline{k}(x) : X \to Y, \; \overline{k}(x)(z) := k(\overline{x}(z), x) \) is continuous. This is because \( \overline{k}(x) = k \circ \overline{i}_x \) where \( i_x : X \to \mathcal{X} \times X \) is the inclusion \( i_x(z) = (\overline{x}(z), x) \; \forall z \in X \). We thus have a map 

\[
\overline{\cdot} : C^0(\mathcal{X} \times X, Y) \to \text{Hom}_{\mathcal{Z}}(\mathcal{Z}, C^0(\mathcal{X}, Y)), \; k \mapsto \overline{k}.
\]

The next lemma shows that the image of \( \overline{\cdot} \) lands in \( C^0(\mathcal{Z}, C^0(\mathcal{X}, Y)) \subseteq \text{Hom}_{\mathcal{Z}}(\mathcal{Z}, C^0(\mathcal{X}, Y)).

**Lemma 28.3** Let \( \mathcal{X}, \mathcal{Y}, \mathcal{Z} \) be three spaces. Then \( \forall k \in C^0(\mathcal{X} \times \mathcal{X}, \mathcal{Y}) \) the map 

\[
\overline{k} : \mathcal{Z} \to C^0(\mathcal{X}, \mathcal{Y}), \; z \mapsto \overline{k}(z) = k(\overline{z}, \cdot)
\]

is continuous, hence we have a map 

\[
\overline{\cdot} : C^0(\mathcal{Z} \times \mathcal{X}, \mathcal{Y}) \to C^0(Z, C^0(\mathcal{X}, \mathcal{Y})).
\]

**Proof:** Enough to show: \( \forall K \subseteq \mathcal{X} \) compact, \( \forall U \subseteq \mathcal{Y} \) open, \( \overline{k}^{-1}(M(K, U)) \) is open in \( \mathcal{Z} \).

Let \( z \in \overline{k}^{-1}(M(K, U)) \). Then \( k(\overline{z} \times K) = \overline{k}(z)(K) \subseteq U \Rightarrow i_z \times K \subseteq k^{-1}(U). \]

Since \( k \) is continuous, \( k^{-1}(U) \) is open in \( \mathcal{X} \times \mathcal{X} \). By Tube Lemma, \( \exists \) open nbhd \( W \) of \( z \) s.t. \( W \times K \subseteq k^{-1}(U). \Rightarrow \forall w \in W \Rightarrow i_w \times K \subseteq k^{-1}(U) \) \( \Rightarrow \overline{k}(W) \subseteq M(K, U). \)

\[
\forall w \in W \Rightarrow W \subseteq (\overline{k})^{-1}(M(K, U)) \Rightarrow \overline{k}^{-1}(M(K, U)) \text{ is open in } \mathcal{Z}.
\]

**Recap:** If \( X \) is LCH, \( \mathcal{X}, \mathcal{Z} \) are spaces we have maps 

\[\overline{\cdot} : C^0(\mathcal{Z}, C^0(\mathcal{X}, \mathcal{Y})) \to C^0(\mathcal{Z} \times \mathcal{X}, \mathcal{Y}), \; k \mapsto \overline{k}, \; \overline{k}(z, x) = (\overline{k}(z))(x) = k(z, x) \in \mathcal{Y} \]

\[\overline{\cdot} : C^0(\mathcal{Z} \times \mathcal{X}, \mathcal{Y}) \to C^0(\mathcal{Z}, C^0(\mathcal{X}, \mathcal{Y})), \; k \mapsto \overline{k}, \; \overline{k}(z) = \overline{k}(z, \cdot), \; \forall z \in \mathcal{Z}.
\]

It's easy to check that the two maps are inverses of each other.
We'd like to prove:

**Theorem 28.4.** Suppose $X, Z$ are LCH, $Y$ Hausdorff. Then the bijection

$$C^o(Z \times X, Y) \leftrightarrow C^o(Z, C^o(X, Y))$$

is a homeomorphism.

To prove 28.4 we need:

**Lemma 28.5.** Let $P$ be an LCH space, $Q$ a space, $B$ a subbasis for a topology on $Q$. Suppose $K = \{ K \subseteq P \text{ compact} \}$ $K$ is a nbd of some $z \in P$. Then

$$\Delta = \{ M(K, U) \mid K \in K, U \in B \}$$

is a subbasis for the compact-open topology on $C^o(P, Q)$.

**Proof.** Enough to show: $\forall K \subseteq P$ compact, $\forall U \subseteq Q$ open, $\forall f \in M(K, U)$, $\exists U_1, \ldots, U_n \in B$

$K_{i_1} \ldots K_{i_n} \in K$ s.t.

$$\bigcap_{i=1}^n M(K_{i}, U_{i}) \subseteq M(K, U).$$

For each $x \in K$ $\exists U_x \in B$ with $f(x) \in U_x \subseteq U$ and a compact nbd $K_x$ of $x$ with $K_x \subseteq f^{-1}(U_x)$. 

(Note that $f \in M(K_x, U_x)$ and $K_x \subseteq K$).

Since $K$ is compact $\exists x_1, \ldots, x_n \in P$ s.t. $K \subseteq K_{x_1} \ldots \cup K_{x_n}$. 

$K_{i_1} \ldots K_{i_n} \subseteq \bigcap_{i=1}^n M(K_{i}, U_{i})$.

$g(K) \subseteq g(U K_{x_i}) = \bigcup_{i=1}^n g(K_{x_i}) \subseteq U$ $U_{x_i} \in U_i$.

$\bigcap_{i=1}^n M(K_{x_i}, U_{x_i}) \subseteq M(K, U)$. \hfill \Box$

**Proof of 28.4.** Consider

$$K = \{ K_1 \times K_2 \subseteq Z \times X \mid K_1 \text{ is a compact nbd of some } z \in Z, K_2 \text{ is a compact nbd of some } x \in X \}$$

By 38.5, $\{ M(K_1 \times K_2, U) \mid K_1, K_2 \in K, U \subseteq Q \text{ open} \}$ is a subbasis for the compact-open topology on $C^o(Z \times X, Y)$. 

By 38.5, $\{ M(K_1, M(K_2, U)) \mid K_1, K_2 \in K, U \subseteq Q \text{ open} \}$ is a subbasis for the compact-open topology on $C^o(Z, C^o(X, Y))$.

Now $k \in M(K_1 \times K_2, U) \iff k(K_1 \times K_2) \subseteq U \iff (\overline{k(K_1)})(K_2) \subseteq U \iff \overline{k(K_1)} \subseteq M(K_2, U) \iff k \in M(K_1, M(K_2, U)).$

Hence the bijection $- : C^o(Z \times X, Y) \rightarrow C^o(Z, C^o(X, Y))$ takes a subbasis of the topology to a subbasis. $- \rightarrow -$ is a homeomorphism. \hfill \Box
Compact-open topology and pullbacks/restrictions.

Lemma 28.6 Let \( f: X' \to X \) be a continuous map and \( Y \) a space. \( \forall f \in C^0(X,Y), \) 
\[
\Psi^* (f) := f \circ \Psi \in C^0(X',Y). \]
The map 
\[
\Psi^*: C^0(X,Y) \to C^0(X',Y) \text{ is continuous (w.r.t. the compact-open topologies).}
\]

Proof. \( X' \) compact and \( Y \) \( U \in Y \) open, 
\[
(\Psi^*)^{-1} (M(K,U)) = \{ f \circ \Psi \in C^0(X,Y) \mid \Psi^* (f) \in M(K,U) \} = \{ f \mid (f \circ \Psi) (K) \in U \}
\]
\[
= \{ f \mid f (\Psi(K)) \in U \} = M(K,\Psi(K),U).
\]
(Note that since \( \Psi \) is continuous and \( K \) is compact, \( \Psi(K) \) is compact.)
\( \Rightarrow \) \( \Psi^* \) is continuous.

Note that if \( X' \subseteq X \) is a subspace, then the map \( C^0(X,Y) \to C^0(X',Y), f \mapsto f|_{X'} \) is continuous since the inclusion map \( i: X' \to X \) is continuous and \( f|_{X'} = f \circ i. \)

Next, we'd like to relate compact-open topology to the topology of uniform convergence on compact sets.

Let \( X \) be a compact space, \( (Y,d) \) a metric space. Then for all \( f,g \in C^0(X,Y) \) the function \( x \mapsto d(f(x),g(x)) \) is continuous (since \( d: Y \times Y \to [0,\infty) \) and \( (f,g): X \to Y \times Y \) are continuous).

Since \( X \) is compact, the set \( \{ d(f(x),g(x)) \mid x \in X \} \) is compact.

(a) \( d_\infty (f,g) := \sup \{ d(f(x),g(x)) \mid x \in X \} \) exist.

Lemma 29.1 \( d_\infty: C^0(X,Y) \times C^0(X,Y) \to [0,\infty) \), \( d_\infty (f,g) = \sup \{ d(f(x),g(x)) \mid x \in X \} \)

is a metric.

Proof (sketch) we check the triangle inequality: \( \forall f,g,h \in C^0(X,Y), \forall x \in X \)
\[
d(f(x),g(x)) \leq d(f(x),h(x)) + d(h(x),g(x)) \leq d_\infty (f,h) + d_\infty (h,g).
\]
\[
\Rightarrow d_\infty (f,g) = \sup \{ d(f(x),g(x)) \mid x \in X \} \leq d_\infty (f,h) + d_\infty (h,g).
\]
The other two checks are even easier.

□
Definition Let \( X \) be a LCH space, \((Y,d)\) a metric space. A net \( \{ f_\alpha \}_\alpha \) in \( C^0(X,Y) \) converges to \( f \in C^0(X,Y) \) uniformly on compact sets if for every \( K \to f|_K \) in \( C^0(K,Y) \) w.r.t. \( d^K: C^0(K,Y) \times C^0(K,Y) \to (0,\infty) \) compact \( K \subseteq X \).

\[
d^K(t',t) = \sup \{ d(f(x),f(x)) | x \in K \} \quad t', t \in C^0(K,Y) \}
\]

We'll show \( f_d \to f \) uniformly on compact sets \( \iff f_d \to f \) in \( C_0 \)-topology on \( C^0(X,Y) \).

Exercise Let \((Y,d)\) be a metric space, \(C \subseteq Y\) compact, \(U \subseteq Y\) open with \( C \subseteq U\). Then \( \exists \epsilon > 0 \) so that \( B_\epsilon(C) = \{ y \in Y | d(y,C) < \epsilon \} \) for some \( C \subseteq Y = \bigcup_{c \in C} B_\epsilon(c) \) is contained in \( U \).

Hint Mimic proof of Lebesgue Lemma.

Lemma 29.2 Let \( X \) be a compact Hausdorff space, \((Y,d)\) a metric space. Then \( (C^0(X,Y), d_\infty) \) is homeomorphic to \( (C^0(X,Y), T_{\text{compact-open}}) \).

Proof Fix \( f \in C_0(X,Y) \) and consider an open ball \( B_\epsilon(f) \subseteq C_0(X,Y) \),

\[ B_\epsilon(f) = \{ g \in C_0(X,Y) | d_\infty(f,g) < \epsilon \} \]

For any \( x \in X \) there exists a compact nbd \( N_x \) with \( N_x \subseteq f^{-1}(B_{\epsilon/3}(f(x))) \).

Since \( X \) is compact, \( \exists K > 0, x_1, \ldots, x_K \in X \) with \( X = N_{x_1} \cup \cdots \cup N_{x_K} \).

Let \( V = \bigcap_{i=1}^K M(N_{x_i}, B_{\epsilon/3}(f(x_i))) \subseteq C_0(X,Y) \) is a nbd of \( f \) in \( T_{\text{compact-open}} \) topology. Suppose \( g \in V \). Then \( \forall x \in N_{x_i}, \quad g(x) \in B_{\epsilon/3}(f(x)) \), i.e.

\[
x \in N_{x_i} \Rightarrow d(g(x),f(x)) < \epsilon/3
\]

\[
\Rightarrow \forall x \in N_{x_i},
\]

\[
d(g(x),f(x)) \leq d(g(x),f(x)) + d(f(x),f(x)) < \epsilon/3 + \epsilon/3 = 2\epsilon/3.
\]

Since \( \bigcup_{i=1}^K N_{x_i} = X \),

\[
d(f(x),g(x)) < \frac{2}{3} \epsilon \quad \forall x \in X.
\]

\[
\Rightarrow d_\infty(f,g) = \sup \{ d(f(x),g(x)) | x \in X \} < \frac{2}{3} \epsilon < \epsilon. \quad \Rightarrow g \in B_\epsilon(f).
\]

\[
\Rightarrow V \subseteq B_\epsilon(f).
\]

Any set in \( T_{\text{d}} \) is in \( T_{\text{compact-open}} \).
Conversely, suppose \( f \in \bigcap_{i=1}^{n} M(K_i, U_i) \) for some \( K_1, \ldots, K_n \subseteq X \) compact, \( U_1, \ldots, U_n \subseteq Y \) open. Then \( f(K_i) \subseteq U_i \ \forall i \). Let \( \varepsilon = \min_{i} \varepsilon_i \). Then \( B_{\varepsilon}(f(K_i)) \subseteq U_i \ \forall i \).

Now if \( g \in B_{\varepsilon}(f) \) then \( d(g(x), f(x)) < \varepsilon \ \forall x \in X \).

\[
\forall i \ \forall x \in X \quad g(x) \in B_{\varepsilon}(f(K_i)) \subseteq U_i \quad \Rightarrow \quad g \in M(K_i, U_i) \ \forall i.
\]

\( \therefore \ B_{\varepsilon}(f) \subseteq \bigcap_{i=1}^{n} M(K_i, U_i) \)

\( \Rightarrow \) Any set in \( T_{\text{comput-open}} \) is in \( T_{\text{def}} \).