Last time:
- 2nd countable + regular → metrizable
- Lindelöf spaces (every cover has a countable subcover)
- 2nd countable → Lindelöf
- Uniform limit of continuous functions is continuous.

Tietze extension theorem

Let $X$ be a normal topological space, $F \subseteq X$ closed subset and $f : F \to [0,1]$ continuous. Then $\exists$ continuous function $\tilde{f} : X \to [0,1]$ with $\tilde{f} | F = f$.

Proof: It's no loss of generality to assume $0 = \inf_{x \in F} f(x), 1 = \sup_{x \in F} f(x)$ (otherwise rescale $f$).

Let $A = f^{-1}([0,\frac{1}{3}])$, $B = f^{-1}([\frac{2}{3},1])$. Then $A, B$ are closed in $F$, disjoint and nonempty. Since $F$ is closed in $X$, $A, B$ are closed in $X$ as well.

By Urysohn's lemma $\exists$ continuous function $g_1 : X \to [0,\frac{1}{3}]$ with $g_1 | A = 0$, $g_1 | B = \frac{1}{3}$.

Then $\forall x \in F$ $f(x) \leq \frac{1}{3} \Rightarrow g_1(x) = 0$ $f(x) \geq \frac{2}{3} \Rightarrow g_1(x) = \frac{1}{3}$.

Let $f_1 = f - g_1 | F$. Then $f_1 : F \to R$ is continuous and $0 \leq f_1(x) \leq \frac{2}{3}$.

Now repeat the construction with $f$ replaced by $f_1$. We get $g_2 \in C^0(X, [0, \frac{1}{3} : \frac{2}{3}])$ so that $f_1(x) \leq \frac{1}{3} : \frac{2}{3} \Rightarrow g_2(x) = 0$ and $f_1(x) \geq \frac{2}{3} : \frac{2}{3} \Rightarrow g_2(x) = \frac{1}{3} : \frac{2}{3}$.

Let $f_2(x) := f_1(x) - g_2(x)$ $\forall x \in F$. Then $0 \leq f_2(x) \leq \frac{2}{3} : \frac{2}{3}$.
Inductive step. Suppose we've constructed \( f_n \in C^0([F, E, (1/3)^n]] \) then \( \exists g_{n+1} \in C^0([0, 1/3 \cdot (2/3)^n]) \) so that \( \forall x \in F \)

\[
\begin{align*}
  f_n(x) &\leq \frac{1}{3} \cdot \left(\frac{2}{3}\right)^n \Rightarrow g_{n+1}(x) = 0 \\
  f_n(x) &> \frac{2}{3} \cdot \left(\frac{2}{3}\right)^n \Rightarrow g_{n+1}(x) = \frac{1}{3} \cdot \left(\frac{2}{3}\right)^n
\end{align*}
\]

Let \( f_{n+1} = f_n - g_{n+1} \mid F \). Then \( 0 \leq f_{n+1}(x) \leq \left(\frac{2}{3}\right)^{n+1} \).

Since \( 0 \leq g_n(x) = \frac{1}{3} \cdot \left(\frac{2}{3}\right)^n \) the series \( g(x) = \sum_{n=1}^{\infty} g_n(x) \) converges uniformly on \( X \).

By 18.3, \( g(x) \) is continuous on \( X \).

Also \( f_n(x) = \sum_{i=1}^{n} g_i(x) \)

\[
\begin{align*}
  f_n(x) - g_n(x) &= f_n(x) \\
  f_n(x) - \sum_{i=1}^{n} g_i(x) &= f_n(x) \\
  f_n(x) - \sum_{i=1}^{\infty} g_i(x) &= f_n(x)
\end{align*}
\]

Since \( 0 \leq f_n(x) \leq \left(\frac{2}{3}\right)^n \), \( 0 = \lim_{n \to \infty} f_n(x) \)

\[
\lim_{n \to \infty} \left( f_n(x) - \sum_{i=1}^{\infty} g_i(x) \right) = f(x) - g(x), \quad \forall x \in F
\]

\[
\begin{align*}
  f(x) &= \frac{1}{3} \cdot \left(\frac{2}{3}\right)^n = \frac{1}{3} \cdot \frac{1}{\frac{2}{3} \cdot (2/3)^n} = 1 \\
  g(x) &= X \to [0, 1] \text{ is the desired function}
\end{align*}
\]
"Application": Moore plane is not normal

Recall \( T = \{(x, y) \in \mathbb{R}^2 \mid y > 0 \} \cup B \implies (x, y) \)
\[ \mathcal{T} = \left\{ \text{standard } \cup \{ \left( x, y \right) \times (0, 1) \} \right\} \]

Let \( L = \{(x, y) \mid y = 0\} \subset \mathcal{T} \).

The subspace topology on \( L \) is discrete.

⇒ any function \( f: L \to [0, 1] \) is continuous.

⇒ \( C^0(L, [0, 1]) = [0, 1]^L \)

If \( \mathcal{T} \) is normal then by Tietze extension theorem

⇒ \( f \in C^0(L, [0, 1]) \) exists \( f' \in C^0(\mathcal{T}, [0, 1]) \) with \( f' |_L = f \).

⇒ \( f \in C^0(L, [0, 1]) \) choose an extension \( f' \).

This gives us an injective map \( C^0(L, [0, 1]) \to C^0(\mathcal{T}, [0, 1]) \)

On the other hand \( \mathbb{Q}^2 \cap \mathcal{T} \) is countable and dense in \( \mathcal{T} \).

⇒ if \( f, g \in C^0(\mathcal{T}, [0, 1]) \) and \( f |_{\mathbb{Q}^2 \cap \mathcal{T}} = g |_{\mathbb{Q}^2 \cap \mathcal{T}} \)

Then \( f = g \).

⇒ \( C^0(\mathcal{T}, [0, 1]) \to C^0(\mathbb{Q}^2 \cap \mathcal{T}, [0, 1]) \) is injective

and \( |[0, 1]|^{\mathbb{Q}^2} \cap \mathcal{T} | = |[0, 1]|^{\mathbb{Q}^2} \cap \mathcal{T} | \)

⇒ \( |C^0(\mathcal{T}, [0, 1])| \leq |[0, 1]|^{\mathbb{Q}^2} \cap \mathcal{T} | \)

⇒ \( |[0, 1] |^{\mathbb{Q}^2} \cap \mathcal{T} | \leq |[0, 1]|^{\mathbb{Q}^2} \cap \mathcal{T} | \)

⇒ \( [0, 1] \cap \mathcal{T} \) is compact, contradiction.

Local compactness

Def: A topological space is locally compact if every point has a compact neighborhood.

Ex.1: Any compact space is locally compact.

Ex.2: \( \mathbb{R}^n \times [0, 1] \), \( U \) open, is locally compact.
**Proof**

Let $U$ be a compact Hausdorff space, $x \in U$. It's enough to find a closed nbd $N$ of $x$ in $Y$ with $N \subseteq U$.

Since $Y$ is compact Hausdorff, it's regular. $\Rightarrow$ there are open sets $O$ and $W \subseteq Y$ with $x \in O$, $W \cap U = \emptyset$. Now let $N = Y \setminus W$. $N$ is closed. Since $O \cap W = \emptyset$, $O \subseteq Y \setminus W$. Also $Y \setminus W \subseteq Y \setminus (Y \setminus U) = U$.

$\therefore$ $N$ is a desired compact nbd of $x$ in $U$.

We'll see that the converse is true as well: if $X$ is LCH then $X$ is homeomorphic to an open subset of a compact Hausdorff space.