Last time: A is a subset of a topological space $X$.

- The closure $\overline{A}$ of $A$ = smallest closed set containing $A$
- The interior $A^o$ of $A$ = largest open set contained in $A$
- The boundary (frontier) $\partial A$ of $A$ = $\overline{A} \cap \overline{X}\setminus A$.
- Neighborhood basis; a space is first countable if every point has a countable neighborhood basis.
- If $X$ is 1st countable, $A \subseteq X$, $x \in \overline{A}$ then $\exists \, x_n \in A$ with $x_n \rightarrow y$.

This is not true in general. So we need a generalization of sequences.

**Definition 9.1** A preorder on a set $\Lambda$ is a relation $\prec$ on $\Lambda$ (ie $\prec \subseteq \Lambda \times \Lambda$ and we write $\lambda_1 \prec \lambda_2$ if $(\lambda_1, \lambda_2) \in \prec$) which is

1) reflexive: $\lambda \prec \lambda$ and
2) transitive: $\lambda_1 \prec \lambda_2$ and $\lambda_2 \prec \lambda_3 \Rightarrow \lambda_1 \prec \lambda_3$.

**Remark** In a preorder $\lambda_1 \prec \lambda_2$ and $\lambda_2 \prec \lambda_1$ does not necessarily imply that $\lambda_1 = \lambda_2$.

If it does, the preorder is called a poset (partially ordered set).

**Definition 9.2** A directed set is a set $\Lambda$ with a preorder $\prec$ so that

$\forall \lambda_1, \lambda_2 \in \Lambda \exists \lambda_3 \in \Lambda$ with $\lambda_1 \prec \lambda_3$ and $\lambda_2 \prec \lambda_3$.

**Example** $(\mathbb{N}, \leq)$ is a directed set.

**Example** Let $X$ be a topological space, $x \in X$ a point, $\Lambda = \{ N \in \mathcal{X} \mid N \text{ is a nbhd of } x \}$. Define a relation $\preceq$ on $\Lambda$ to be reverse inclusion: $N_1 \preceq N_2 \iff N_1 \supseteq N_2$.

Then $\forall N_1, N_2 \in \Lambda$, $N_1 \cap N_2 \neq \emptyset$ since $x \in N_1 \cap N_2$ and $N_1 \preceq N_1 \cap N_2$, $N_2 \preceq N_1 \cap N_2$.

$\Rightarrow \Lambda$ is a directed set.

**Example** Fix an interval $[a,b]$. The set of partitions of $[a,b]$ is a directed set.
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$P_1 < P_2 \iff P_2$ refines $P_1$. And any two partitions have a common refinement.

**Definition 9.3** A net $x : \Lambda \to X$, $\lambda \mapsto x_\lambda$, in a topological space $X$ is a function where $\Lambda$ is a directed set. We denote a net $x : \Lambda \to X$ by $(x_\lambda)_{\lambda \in \Lambda}$.

**Definition 9.4** A net $(x_\lambda)_{\lambda \in \Lambda}$ in a topological space $X$ converges to $y \in X$ if a neighborhood $W$ of $y$ $\exists \lambda_0 \in \Lambda$ so that $\lambda_0 \leq \lambda \Rightarrow x_\lambda \in W$.

**Notation** $x_\lambda \to y$

If $x_\lambda \to y$ we say that $y$ is a limit of $(x_\lambda)_{\lambda \in \Lambda}$.

A net $(x_\lambda)_{\lambda \in \Lambda}$ is convergent if it has a limit.

**Proposition 9.5** Let $X$ be a topological space, $A \subseteq X$ a subset. $y \in \overline{A} \iff \exists$ a net $(x_\lambda)_{\lambda \in \Lambda}$ in $A$ with $x_\lambda \to y$.

**Proof** ($\Leftarrow$) Suppose $(x_\lambda)_{\lambda \in \Lambda} \subseteq A$ is a net and $x_\lambda \to y$. Then since $x_\lambda \to y$ a nbd $W$

$\forall x_\lambda | x_\lambda \in \Lambda \exists W \cap A = \emptyset$. By Proposition 7.3, $y \in \overline{A}$.

($\Rightarrow$) Suppose $y \in \overline{A}$. Let $\Lambda$ be the set of all neighborhoods of $y$.

$\Lambda$ is directed by reverse inclusion. By 7.3, $\forall N \in \Lambda$, $N \cap A \neq \emptyset$.

For each $N \in \Lambda$ choose $x_N \in A \cap N$.

Suppose we are given a nbd $W$ of $y$. If $W < N$ then $N \subseteq W$. So $x_N \in W$ and $x_N \in A$ with $W < N$. $x_N \to y$.

By construction $x_N \in A$ for $N$.

By construction $x_N \to y$.

**Nets are useful for checking continuity.**

**Proposition 9.6** A map $f : X \to Y$ between two topological spaces is continuous $\iff$

a net $(x_\lambda)_{\lambda \in \Lambda}$ in $X$ with $x_\lambda \to w$ in $X$, $f(x_\lambda) \to f(w)$ in $Y$. 
Proof (⇒) Suppose $f$ is continuous, $(x_\lambda)_{\lambda \in \Lambda}$ a net in $X$ with $x_\lambda \to w$. Let $U$ be a nbd of $f(w)$ in $Y$. Since $f$ is continuous, $f^{-1}(U)$ is an nbd of $w$ in $X$. Since $x_\lambda \to w$, $\exists \lambda_0 \in \Lambda$ so that $\lambda_0 < \lambda \Rightarrow x_\lambda \in f^{-1}(U) \Rightarrow f(x_\lambda) \in U$. $\therefore f(x_\lambda) \to f(w)$.

(⇐) Suppose $f$ is not continuous. Then $\exists V \subseteq Y$ open so that $K := f^{-1}(V)$ is not open in $X$. Since $K$ is not open, $K \neq K^\circ$. Choose $w \in K \setminus K^\circ$. Let $\Lambda$ be the set of open nbds of $w$ directed by reverse inclusion: $U_1 \subset U_2 \Rightarrow U_1 \supset U_2$. Since $w \in K \setminus K^\circ$, for any open nbhd $U$ of $w$, $U \setminus K \neq \emptyset$ (otherwise $w \in K^\circ$)

For each $U \in \Lambda$ choose $x_U \in U \setminus K = U \setminus f^{-1}(V)$. Then $f(x_U) \notin V$. Since $w \in K = f^{-1}(V)$, $f(w) \in V$. Therefore $f(x_U) \to f(w)$.

It remains to show that $x_U \to w$. Let $N$ be an nbd of $w$. Then $N^\circ$ is an open nbhd of $w \Rightarrow N^\circ \subseteq \Lambda$. If $U \in \Lambda$ and $N^\circ \subset U$ then $N^\circ \subset U_2 \Rightarrow x_U \in U \subseteq N$ for all $U$ with $N^\circ \subset U$. $\therefore x_U \to w$. Thus if $f : X \to Y$ is not continuous $\exists$ a net $(x_\lambda)_{\lambda \in \Lambda}$ in $X$ with $x_\lambda \to w$ and $f(x_\lambda) \to f(w)$.

(□)

Proposition 9.7 A topological space $X$ is Hausdorff $\iff$ limits of nets in $X$ are unique.

Recall: $X$ is Hausdorff $\iff \forall x, y \in X$ with $x \neq y \exists$ open nbds $U$ of $x$, $V$ of $y$ with $U \cap V = \emptyset$.

Proof of 9.7 Suppose $X$ is Hausdorff, $(x_\lambda)_{\lambda \in \Lambda}$ a net in $X$ with $x_\lambda \to a$ and $x_\lambda \to b$ in $X$. If $a \neq b$, $\exists$ open nbds $U$ of $a$, $V$ of $b$ s.t. $U \cap V = \emptyset$.

Since $x_\lambda \to a \Rightarrow \exists \lambda_1 \in \Lambda$ so that $\lambda_1 < \lambda \Rightarrow x_\lambda \in U$.

Since $x_\lambda \to b \Rightarrow \exists \lambda_2 \in \Lambda$ so that $\lambda_2 < \lambda \Rightarrow x_\lambda \in V$.

Since $\Lambda$ is directed, $\exists \lambda_3 \in \Lambda$ with $\lambda_1 < \lambda_3$, $\lambda_2 < \lambda_3 \Rightarrow x_\lambda \in U \cap V$, which contradicts $U \cap V = \emptyset$. $\therefore a = b$.

(⇐) Suppose $X$ is not Hausdorff. Then $\exists a, b \in X$ s.t. $a \neq b$ and for any nbhd $U$ of $a$ and any nbhd $V$ of $b$, $U \cap V \neq \emptyset$.

We now argue: $\exists$ a net $(x_\lambda)_{\lambda \in \Lambda}$ in $X$ with $x_\lambda \to a$ and $x_\lambda \to b$. 

(□)
Let \( \Lambda = \{(U,V) | U \text{ a nbd of } a, V \text{ a nbd of } b \text{ and } U \cap V \neq \emptyset \} \).

Define a relation \( \prec \) on \( \Lambda \) by \( (U,V) \prec (U',V') \iff (U \cap U' \text{ and } V \cap V') \).

It's not hard to see that \( (\Lambda, \prec) \) is a directed set.

Now for each \((U,V) \in \Lambda \) pick \( x_{U,V} \in V \setminus U \neq 0 \).

If \( (U,V) \prec (U',V') \) then \( U' \subset U \text{ and } V' \subset V \Rightarrow x_{U,V} \in V \setminus U \subset U \Rightarrow x_{U,V} \prec a \).

Similarly \( x_{U,V} \prec b \).

The following characterization of Hausdorff spaces will be useful:

**Proposition 9.8** A space \( X \) is Hausdorff \( \iff \) the diagonal \( \Delta_X = \{(x,y) \in X \times X | x = y\} \) is closed in \( X \times X \) (where \( X \times X \) has product topology).

**Proof** For \( x,y \in X \), \( x \neq y \iff (x,y) \notin \Delta_X \).

\( \Delta_X \in X \times X \) is closed \( \iff \) \( X \times X \setminus \Delta_X \) is open

\( \iff \forall (x,y) \in X \times X \setminus \Delta_X \exists \text{ an open nbd } W \text{ of } (x,y) \text{ with } W \cap \Delta_X = \emptyset \).

The sets of the form \( U \times V, U \setminus V \in X \) are open, form a basis of the product topology of \( X \times X \). Thus

\( X \times X \setminus \Delta_X \) is open \( \iff \forall x,y \in X \text{ with } x \neq y \exists \text{ open nbd } U \times V \text{ of } (x,y) \text{ with } (U \times V) \cap \Delta_X = \emptyset \).

Finally note: \( (U \times V) \cap \Delta_X \neq \emptyset \iff U \cap V = \emptyset \).