Last time, Quotient topology: \((X, T_x)\) topological space, \(\sim\) equivalence relation on \(X\), \(q: X \to X/\sim\) quotient map. The quotient topology \(T_{\text{quot}}\) on \(X/\sim\) is the largest topology so that \(q: X \to X/\sim\) is continuous:
\[ U \in T_{\text{quot}} \iff q^{-1}(U) \in T_x. \]

- If \(f: (X, T_x) \to (Y, T_Y)\) is a continuous surjective map and \(T_Y\) is the largest topology so that \(f\) is continuous (i.e. \(f^{-1}(U)\) open \(\Rightarrow U \subseteq Y\) is open) then \((Y, T_Y)\) is homeomorphic to \((X/\sim, T_{\text{quot}})\) where \(x \sim x' \iff f(x) = f(x')\).

- Limit points: \((X, T)\) topological space, \(A \subseteq X\) a subset. \(x \in X\) is a limit point of \(A\) \iff for any neighborhood \(N\) of \(x\), \(N \cap (A \setminus \{x\}) \neq \emptyset\).

Recall, A subset \(C\) of a topological space \(X\) is closed iff \(X \setminus C\) is open.

**Definition 7.1** Let \(A\) be a subset of a topological space \(X\). The closure \(\overline{A}\) of \(A\) is the smallest closed subset of \(X\) containing \(A\):
\[ \text{if } C \subseteq X \text{ is closed and } A \subseteq C \text{ then } \overline{A} \subseteq C. \]

**Lemma 7.2** The closure \(\overline{A}\) of a subset \(A\) of a topological space \(X\) exists and is unique.

**Proof** The uniqueness of \(\overline{A}\) is easy since there is only one smallest closed set containing \(A\).

**Existence**

Let \(\overline{A} := \bigcap_{\substack{C \subseteq X \text{ closed} \\text{ and } A \subseteq C}} C\). Then

(i) \(\overline{A}\) is closed (why?)

(ii) \(A \subseteq \overline{A}\)

(iii) if \(C' \subseteq X\) is closed and \(A \subseteq C'\) then \(\overline{A} = \bigcap_{A \subseteq C' \text{ closed}} C'\).

**Remark** \(A = \overline{A} \iff A\) is closed (why?)

**Proposition 7.3** Let \(X\) be a topological space, \(A \subseteq X\). Then
\[ \overline{A} = A \cup A' \]
(\(A'\) limit points, \(A\) closure)
In particular, \( x \in A \Rightarrow \exists \text{ nbd} N \text{ of } x \) \( N \cap A \neq \emptyset \).

Proof. Note first:

\[ x \notin A \cup A' \Leftrightarrow \exists \text{ nbd} N \text{ of } x \text{ so that } N \cap A = \emptyset. \]

We now argue: \( x \notin A \Rightarrow \exists \text{ nbd} N \text{ of } x \text{ with } N \cap A = \emptyset. \)

(\implies) If \( x \notin A \), \( x \in X \setminus A \). Since \( A \) is closed, \( N = X \setminus A \) is open.

This \( N \) is a nbd of \( x \) with \( N \cap A = \emptyset \). Since \( A \subseteq A \), \( N \cap A = \emptyset \).

(\(\Leftarrow\)) Suppose \( \exists \text{ nbd} N \text{ of } x \text{ with } N \cap A = \emptyset \). Then \( \exists U \subseteq X \text{ open with } x \in U \subseteq N \). Since \( N \cap A = \emptyset \), \( U \cap A = \emptyset \) as well \( \Rightarrow A \subseteq X \setminus U \), which is closed.

\( \Rightarrow A \subseteq X \setminus U \). But \( x \in U \Rightarrow x \notin A \).

**Definition 7.4.** Let \((X, T_X)\) be a topological space, \( \{x_n\} \subseteq X \) a sequence (i.e. a function \( \mathbb{N} \rightarrow X \), \( n \rightarrow x_n \)). The sequence \( \{x_n\} \) converges to \( y \in X \) if \( \forall \text{ nbd } W \text{ of } y \exists N \in \mathbb{N} \text{ so that } x_n \in W \text{ for all } n \geq N \). We say: \( y \) is a limit of \( \{x_n\} \), \( \{x_n\} \) converges to \( y \) and write \( x_n \rightarrow y \).

**Remark.** You may remember from your analysis classes:

- Limits of sequences in \( \mathbb{R}^n \) are unique.
- For a subset \( A \subseteq \mathbb{R}^n \), \( y \in A \Leftrightarrow \exists \text{ a sequence } \{x_n\} \subseteq A \text{ with } x_n \rightarrow y \).

Both statements are false for sequences in general topological spaces.

**Example.** \( X = \{a, b, c\} \), three point space. \( T_X = \{ \emptyset, X, \{a, b\}, \{c, b\}, \{b, c\} \} \).

Consider the constant sequence \( x_n = b \forall n \).

Then \( x_n \rightarrow b \), obviously. But \( x_n \rightarrow a \) as well: the neighborhoods of \( a \) are \( \{a, b\} \) and \( X \) and \( x_n \in \{a, b\} \forall n \). Even worse \( x_n \rightarrow c \) (check it).

However, the following result is true.

**Proposition 7.5.** Let \( X \) be a topological space, \( A \subseteq X \), \( \{x_n\} \subseteq A \) a sequence...
with $x_n \to y$. Then $y \in \overline{A}$.

**Proof.** Since $x_n \to y$, for any nbhd $W$ of $y$ $\exists N$ st $x_N \in W$. Since $x_N \in A$, $W \cap A \neq \emptyset$.

By Proposition 7.3, $y \in \overline{A}$.

**Example.** Consider a countable family of sets $\{Y_i\}_{i \in \mathbb{N}}$ with $Y_i = (0, \infty)$. Then the product $\prod_{i=1}^{\infty} Y_i$ is the set $\mathbb{R}^\mathbb{N}$ of sequences of real numbers $(x_n)_{n \in \mathbb{N}}$. Consider $A = \{ (x_n) \in \mathbb{R}^\mathbb{N} : x_n > 0 \text{ for all } n \}$.

Let $0$ denote the constant zero sequence: $(0)_n = 0$ for all $n$. Give $\mathbb{R}^\mathbb{N}$ the box topology $T_{\text{box}}$: it's a topology generated by $B_{\text{box}} = \{ \prod_{i=1}^{\infty} U_i : U_i \in \mathbb{R} \text{ open} \}$.

Let $W$ be a neighborhood of $0$ in $(\mathbb{R}^\mathbb{N}, T_{\text{box}})$. Then $\exists \prod_{i=1}^{\infty} U_i \in B_{\text{box}}$ so that $0 \in \prod_{i=1}^{\infty} U_i \subseteq W$. Since $0 \in \prod_{i=1}^{\infty} U_i$, $0 \notin U_i \forall i$. Since $U_i \in \mathbb{R}$ is open and $0 \notin U_i$ for all $i$, $\exists \varepsilon > 0$ so that $(-\varepsilon, \varepsilon) \subseteq U_i \Rightarrow U_i \cap (0, \infty) \neq \emptyset$.

\[
\prod_{i=1}^{\infty} U_i \cap A \neq \emptyset \Rightarrow W \cap A \neq \emptyset \Rightarrow 0 \in \overline{A}.
\]

We now argue that no sequence $(a_n)_{n \in \mathbb{N}}$ in $A$ converges to $0$. Pick a sequence $(a_n)_{n \in \mathbb{N}}$ in $A$. Each $a_n \in \mathbb{R}^{\mathbb{N}}$, so each $a_n$ is a sequence $(a_n^k)_{k \in \mathbb{N}}$ in $\mathbb{R}$. And since $a_n \in A$, $a_n^k > 0$ for all $k$. Consider $U = \prod_{k=1}^{\infty} (-a_n^k, a_n^k)$. Since $a_n^k \notin (-a_n^k, a_n^k)$ for all $k$, $a_n \notin U$. But $U$ is a neighborhood of $0$. \Rightarrow $a_n \not\to 0$.

Thus $0 \in \overline{A}$ but there is no sequence $(a_n)_{n \in \mathbb{N}}$ in $A$ with $a_n \to 0$.

There is a nice class of topological spaces where limits of sequences are unique.

**Definition 7.6.** A topological space $X$ is Hausdorff (also called $T_2$) if $\forall x, y \in X$ with $x \neq y$ $\exists$ open sets $U, V \subseteq X$ with $x \in U$, $y \in V$ and $U \cap V = \emptyset$. 

Example We have seen that any metric space \((X,d)\) is Hausdorff:
\[ \forall x,y \in X \text{ with } x \neq y, \quad r = d(x,y) > 0, \quad \text{and } \, B_{r/2}(x) \cap B_{r/2}(y) = \emptyset. \]

Nonexample \(X = \{ a, b, c \} \) with \( T_x = \emptyset, \{ a, b \}, \{ a, b, c \} \) is not Hausdorff:
for any open nbhd \( U \) of \( a \) and any open nbhd \( V \) of \( c \), \( U \cap V \supset b \).

Nonexample \( \mathbb{R} \) with cofinal topology \((\forall U \subseteq \mathbb{R} \text{ open} \iff \mathbb{R} \setminus U \text{ is finite})\) is not Hausdorff.

**Proposition 7.7** In a Hausdorff topological space limits of sequences are unique (whenever they exist).

**Proof** Suppose \( X \) is Hausdorff, \( \{ x_n \} \subseteq X \) a sequence with \( x_n \to y \) and \( x_n \to z \).
If \( y \neq z \) \( \exists \) open nbds \( U \) of \( y \), \( V \) of \( z \) s.t. \( U \cap V = \emptyset. \) Since \( x_n \to y \) \( \exists N \in \mathbb{N} \)
so that for \( n \geq N \), \( x_n \in U \). Since \( U \cap V = \emptyset \), \( x_n \notin V \) for \( n \geq N \) \( \Rightarrow x_n \to z. \)

Contradiction. Hence \( y = z. \)

There is a class of spaces for which \( y \in A \iff \exists \text{ a sequence } \{ x_n \} \subseteq A \text{ with } x_n \to y. \)
They are called \( 1^{st} \) countable spaces.

For more general spaces one replaces sequences with nets; nets are
"sequences with possibly uncountable indexing sets."