Recall that if $X$ is an LCH space then we can give $X^+ = X \cup \{x_0\}$ a topology so that 

1. $X^+$ is compact and 
2. $X \to X^+ \{x_0\}$ is an embedding.

Moreover if $X$ is not compact then $i(X)$ is dense in $X^+$ and therefore $i : X \to X^+$ is a compactification of $X$, a 1-point compactification.

A given space may have many compactifications and $i : X \to X^+$ is the smallest: $X^+$ has only one extra point compare to $X$.

There is also the biggest — the Stone–Čech compactification.

40.1 Recall that for any set $M$ we have the product $[0,1]^M$ of $M$ copies of $[0,1]$. $[0,1]^M = \{x : M \to [0,1]\}$, the set of all functions from $M$ to $[0,1]$. A function $x : M \to [0,1]$ assigns to each $m \in M$ the number $x_m = x(m) \in [0,1]$. Thus $[0,1]^M = \{(x_m)_{m \in M}\}$, the set of all “$M$-tuples” of numbers in $[0,1]$. The product topology on $[0,1]^M$ is the smallest topology so that the projections

$$\pi_m : [0,1]^M \to [0,1], \quad m \in M, \quad \pi_m(x) = x(m) = x_m$$

are continuous.

We proved that the product topology has a universal property. In the case of $[0,1]^M$ the property amounts to the following:

For any space $Z$ and for any family $\{f_m : Z \to [0,1]\}_{m \in M}$ of continuous functions

$$\exists! \text{ continuous map } f : Z \to [0,1]^M \text{ so that } \pi_m f = f_m \forall m \in M.$$ (equivalently $\forall m \in M, \forall z \in Z, \ (f(z))_m = f_m(z)$)

40.2 A function $\Psi : N \to M$ between two sets induces a function

$$\Psi : [0,1]^M \to [0,1]^N.$$ 

There are three equivalent ways to think of $\Psi$:

1. Since elements of $[0,1]^M$ and $[0,1]^N$ are $[0,1]$-valued functions,

$$\Psi (M \to [0,1]) = x \circ \Psi : N \to [0,1]$$

2. If we think of $[0,1]^M$ as tuples $(x_m)_{m \in M}$ then $\Psi((x_m)_{m \in M}) = (x_m(\Psi(n))_{n \in N} \in [0,1]^M$. 

3. If we think of $[0,1]^M$ as functions $f : M \to [0,1]$ then $\Psi(f) = f \circ \Psi : N \to [0,1]$.
Let $X$ be a space and let $M$ be a subset of the set $C^0(X, [0,1])$. Then for each $x \in X$ we have a function $\text{ev}_x : M \rightarrow [0,1]$. It is defined by

$$\text{(ev}_x(x))(t) = f(t) \quad \text{for all } f \in M.$$ 

Putting the functions $\{\text{ev}_x(x)\}_{x \in X}$ together we get a function

$$\text{ev}_x : X \rightarrow [0,1]^M, \quad (\text{ev}_x(x))(f) = f(x) \quad \forall x \in X, \forall f \in M.$$ 

Equivalently

$$\pi^M_x \circ \text{ev}_x = f$$

for all $f \in M$ (here as before $\pi^M_x : [0,1]^M \rightarrow [0,1]$ are the canonical projections).

Hence $\text{ev}_x : X \rightarrow [0,1]^M$ is continuous.

Note that by Lemma 17.2 if $M \subseteq C^0(X, [0,1])$ is the set of functions that separate points and closed sets and if $X$ is $T_1$ then $\text{ev}_x : X \rightarrow [0,1]^M$ is an embedding.

In particular if $X$ is $T_1$ and completely regular then $C^0(X, [0,1])$ separates points and closed sets, so $\text{ev}_x : X \rightarrow [0,1]^{C^0(X, [0,1])}$ is an embedding.

Since $[0,1]^{C^0(X, [0,1])}$ is compact, $\overline{\text{ev}_x(X)} = [0,1]^{C^0(X, [0,1])}$ is compact.

Let $X$ be a $T_1$ completely regular space and $\text{ev}_x : X \rightarrow [0,1]^{C^0(X, [0,1])}$ the corresponding embedding (see 40.4 above). Let $\beta(x) = \overline{\text{ev}_x(x)}$ and

$\eta_x : X \rightarrow \beta(x)$ the map $\text{ev}_x : X \rightarrow [0,1]^{C^0(X, [0,1])}$ thought of as a map into $\beta(x)$.

The Stone-Čech compactification of $X$ is the embedding $\eta_x : X \rightarrow \beta(x)$. 

Let $X \rightarrow Y$ be a continuous map between spaces. We then have a map
\( \psi^*: C^0(Y, [0,1]) \to C^0(X, [0,1]), \) \( \psi^*(f) := f \circ \psi \) for all \( f \in C^0(Y, [0,1]) \)

By 40.2 we have \( \psi^*: [0,1]^X \to [0,1]^Y \)

**Claim** For any continuous map \( \psi: X \to Y \) the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\psi} & Y \\
\| & \downarrow & \| \\
[0,1]^X & \xrightarrow{\psi^*} & [0,1]^Y \\
\end{array}
\]

commutes.

**Proof** It's enough to show that \( \varphi \in C^0(Y, [0,1]) \)

\[
\pi_h^X \circ \psi = \pi_h^Y \circ \psi^* \circ \psi_x .
\]

By 40.2(3), \( \pi_h^X \circ \psi = \pi_h^X \)

By 40.3 (see 411 in particular) \( \pi_h^Y \circ \psi = h \) and \( \pi_h^X \circ \psi_x = \psi^* h \)

Hence \( \pi_h^Y \circ \psi = \pi_h^X \circ \psi_x = \psi^* h = h \circ \psi = (\pi_h^Y \circ \psi_x) \circ \psi \). \( \square \)

**Lemma 40.6** For any continuous map \( \psi: X \to Y \)

\[
(\psi^*) (\psi_x) \subseteq \psi_y.
\]

**Proof** For any continuous function \( F: Z \to W \) between spaces and for any subspace \( A \subseteq Z \), \( F(A) \subseteq \overline{F(A)} \). Therefore 90.5

\[
\psi^* (\psi_x) \subseteq \psi^* (\psi_x) \subseteq (\psi^* \circ \psi_x)(X) = \psi_y (\psi_x) \subseteq \psi_y (\psi_x).
\]

**Notation** For any topological space \( X \) let \( \beta(X) := \overline{\psi_x} \) (in \( C^0(X, [0,1]) \))

For any continuous map \( \psi: X \to Y \) define \( \beta(X) : \beta(X) \to \beta(Y) \) to be \( \psi^* \mid \beta(X) \).

By lemma 40.6, \( (\psi^*) (\psi_x) \subseteq \psi_y (\psi_x) = \psi_y \), so \( \beta(X) \) is a well-defined continuous map.

For each topological space \( X \) let \( \eta_X : X \to \beta(X) \) be the map \( \psi_x : X \to \psi_x \).

We have observed that if \( X \) is completely regular and \( T_\beta \) then \( \eta_X : X \to \beta(X) \) is an embedding, the Stone-Cech compactification.

In general, \( \beta(X) \subseteq [0,1]^X \) is a compact Hausdorff space for any space \( X \).
It's not hard to show that $\beta$ is a functor from $\text{Top}$, the category of top spaces to $\text{CHTop}$, the category of compact Hausdorff spaces. 40.5 and 40.6 then imply that for any continuous map $\eta: X \to Y$ the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\eta} & Y \\
\downarrow & & \downarrow \\
\beta(X) & \xrightarrow{\beta(\eta)} & \beta(Y)
\end{array}
$$

commutes.

Interpretation: $\eta$ is a natural transformation from $\text{id}_{\text{Top}}: \text{Top} \to \text{Top}$ to $i \circ \beta$ where $i: \text{CHTop} \to \text{Top}$ is the inclusion functor.

40.7 Note that if $X$ is compact Hausdorff then, since $\text{ev}_X: X \to \text{co}(X, [0,1])$ is continuous, $\text{ev}_X(X)$ is closed. Hence $\beta(X) = \text{ev}_X(X)$. Since compact Hausdorff spaces are completely regular, $\text{ev}_X: X \to \text{ev}_X(X) = \beta(X)$ is a homeomorphism.

**Lemma 40.8** Let $X$ be a completely regular $T_1$ space, $p: X \to Y$ a compactification (so $Y$ is compact Hausdorff and $p(X)$ is dense in $Y$). Then there exists a unique continuous map $\tilde{\eta}_X: \beta(X) \to Y$ so that $\beta(X) \xrightarrow{\tilde{\eta}_X} Y$ commutes.

Here as before, $\eta_X: X \to \beta(X)$ is the Stone-Čech compactification.

**Proof** Since $\eta_X(X)$ is dense in $\beta(X)$, $\tilde{\eta}$ has to be unique. We know that the diagram $\beta(X) \xrightarrow{\beta(p)} \beta(Y)$ commutes. Since $Y$ is compact, $\beta(Y) \xrightarrow{\text{id}_{\beta(Y)}} \beta(Y)$ is a homeomorphism. So let $\tilde{\eta} = \eta_Y \circ \beta(p)$. $\square$