Last time, I proved that if a functor $F : C \to D$ is fully faithful and essentially surjective, then $F$ has a weak inverse. That is, there is a functor $G : D \to C$ and natural isomorphisms $\psi : \text{id}_C \cong GF$, $\eta : FG \cong \text{id}_D$.

Hence $F$ is part of an equivalence of categories and $C$ and $D$ are equivalent.

2) defined pushouts: a pushout of $a \xleftarrow{i_1} c \xrightarrow{f_2} b$ in a category $C$ is $p(c)$ and $a \xleftarrow{i_2} p \xrightarrow{f_2} b$ so that (i) $c \to b$ commutes and (ii) the following universal property of $\psi$ holds: given a commutative diagram $c \to b \to k_1 \in C$

3) $k : p \to d$ so that $c \to b \xrightarrow{k_1} d$ commutes.

We have seen: for any three sets and two functions $A \xleftarrow{i_1} C \xrightarrow{f_2} B$ there is a pushout $A \xleftarrow{i_1} P \xrightarrow{i_2} B$. Namely $P = (A \cup B) / \sim$

where ~ "identifies" (declares equivalent) $f_1(x) \in A \leftrightarrow A \cup B$

with $f_2(x) \in B \leftrightarrow A \cup B$ for all $x \in C$.

One says: "arbitrary pushouts exist in the category of sets."

Lemma 33.1 Let $a \xleftarrow{i_1} p \xrightarrow{i_2} b$ and $a \xleftarrow{i_1'} p' \xrightarrow{i_2'} b$ be two pushouts of $a \xleftarrow{f_1} c \xrightarrow{f_2} b$ in a category $C$. Then $\exists!$ isomorphism $\psi : p \to p'$ so that $a \xleftarrow{i_1} p \xrightarrow{i_2} b$ commutes.

Proof By the universal property of $a \xleftarrow{i_1} p \xrightarrow{i_2} b \exists! \psi : p \to p'$ so that $c \xleftarrow{f_1} b$ commutes.

By the same argument $a \xleftarrow{i_1'} p' \xrightarrow{i_2'} b$ commutes. $\exists! \psi : p' \to p$ so that
But \( \text{id}_p \cdot p \cdot p \) also makes (\( \ast \)) commute. By the uniqueness part of the universal property of \( a \cdot p \xrightarrow{\psi_2} b \), \( \psi \circ \psi = \text{id}_p \). Similarly, \( \psi \circ \psi = \text{id}_p \).

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(Important) observation: Let \( X \) be a space, \( \{U_1, U_2\} \) an open cover of \( X \). Then \( U_1 \xrightarrow{i_1} X \xleftarrow{i_2} U_2 \) (where \( i_1, i_2 \) are inclusions) is a pushout of \( U_1 \xleftarrow{f_1} U_1U_2 \xrightarrow{f_2} U_2 \) (where \( f_1, f_2 \) are also inclusions).

Reason. Suppose \( Z \) is a space, \( g_1 : U_1 \rightarrow Z \), \( g_2 : U_2 \rightarrow Z \) two continuous functions with \( g_1 |_{U_1 U_2} = g_2 |_{U_1 U_2} \). Then \( \exists! \) continuous function \( g : X \rightarrow Z \) so that \( g \circ i_1 = g_1 \), \( g \circ i_2 = g_2 \). Namely we define \( g(x) = \frac{1}{z} \cdot g_1(x) \cdot x \in U_1 \).

Since \( g_1 |_{U_1 U_2} = g_2 |_{U_1 U_2} \), \( g \) is well-defined. Since \( U_1, U_2 \) are open and \( g_1, g_2 \) are continuous, \( g \) is continuous.

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**Theorem (Brown - Seifert - van Kampen)** Let \( X \) be a space, \( \{U_1, U_2\} \) an open cover, \( i_1 : U_1 \rightarrow X \), \( i_2 : U_2 \rightarrow X \) inclusions. Then \( \pi_1 U_1 \xrightarrow{\pi_1 i_1} \pi_1 X \xleftarrow{\pi_1 i_2} \pi_1 U_2 \) is a pushout of \( \pi_1 U_1 \xleftarrow{\pi_1 i_1} \pi_1(U_1 U_2) \xrightarrow{\pi_1 i_2} \pi_1 U_2 \), where \( i_1 : U_1 U_2 \rightarrow U_k \), \( k = 1, 2 \) are the inclusions.

**Remark** Since \( U_1 \xrightarrow{i_1} X \xleftarrow{i_2} U_2 \) is a pushout of \( U_1 \xleftarrow{i_1} U_1 U_2 \xrightarrow{i_2} U_2 \) in \( \text{Top} \), \( \text{B-S-vK} \) says: the functor \( \pi_1 : \text{Top} \rightarrow \text{Groupoid} \) takes pushouts to pushouts.

**Theorem (Seifert - van Kampen)** Let \( X \) be a space, \( \{U_1, U_2\} \) an open cover of \( X \) and suppose \( U_1 U_2 \) is path connected. Then \( \pi_1(U_1, x_0) \xrightarrow{\pi_1 i_1} \pi_1(X,x_0) \xleftarrow{\pi_1 i_2} \pi_1(U_2, x_0) \).
is the pushout of \( \pi_1(X, x_0) \leftarrow \pi_1(U_1 \cup U_2, x_0) \rightarrow \pi_1(U_2, x_0) \) in the category Group of groups and homomorphisms.

**Remark** An older name for pushouts in Group is amalgamated free products.

We will deduce Seifert-van Kampen from Brown-Seifert-van Kampen.

**Remarks 33.2** (they will be used in proving B-S-vK and in computing examples.)

1) Let \( X \) be a space and \( \gamma: [a,b] \rightarrow X \) a continuous map, a path. We have a homeomorphism \( Y: [0,1] \rightarrow [a,b] \), \( Y(t) = (1-t)a + tb \). Then \( [Y] \gamma \) is a morphism in \( \Pi X \) from \( \gamma(0) = \gamma(a) \) to \( \gamma(1) = \gamma(b) \). We will informally say that \( \gamma \) represents a morphism in \( \Pi X \) from \( \gamma(a) \) to \( \gamma(b) \) and write \( \gamma(a) \xrightarrow{\gamma} \gamma(b) \) (with \( \gamma \) suppressed).

2) Let \( \mathcal{U} \) be an open cover of a space \( X \) and \( \gamma: [a,b] \rightarrow X \) a continuous map. Then \( \exists \) a partition \( a = t_0 < t_1 < \cdots < t_n = b \) of \( [a,b] \) so that \( \gamma^i \gamma([t_{i-1}, t_i]) \) is contained in some \( U_\alpha \) (\( \alpha \) depends on \( i \)). Reason: Since \( \gamma^{-1}(\mathcal{U}) \) is a cover of \( [a,b] \), Lebesque’s lemma implies that \( \exists \delta > 0 \) so that \( |x - y| < \delta \Rightarrow [x, y] \subseteq \gamma^{-1}(U_\alpha) \) for some \( \alpha \).

3) Suppose \( \gamma: [a,b] \rightarrow X \) is continuous, as before, an \( a = t_0 < t_1 < \cdots < t_n \) a partition of \( [a,b] \). Then \( [Y] \gamma = \gamma([t_{n-1}, t_n]) \cdots \gamma([t_0, t_1]) \) in \( \Pi X \).

**Aside:** Action groupoids.

An action of a group \( G \) on a set \( X \) gives rise to a groupoid. It’s denoted by \( G \times X \) and by \( G \times X \rightarrow X \). \( G \times X \) is defined as follows. The objects of \( G \times X \rightarrow X \) are the elements of \( X \). A morphism in \( G \times X \) is a pair \( (g, x) \in G \times X \). \( (g, x) \) is a morphism from \( x \) to \( g \cdot x \); \( g \cdot x \xleftarrow{(g, x)} x \). (So \( x \) is the source of \( g \cdot x \) and \( g \cdot x \) is the target.) The composition in \( G \times X \) comes from the multiplication in \( G \): 

\[
\begin{align*}
\begin{array}{ccc}
  \bullet & \xleftarrow{(g_2, g_1 x)} & \bullet \\
  g_2 \cdot (g_1 x) & \rightarrow & (g_2, g_1 x) \quad \text{where} \quad (g_2, g_1 x) \circ (g_1, x) = (g_2 g_1, x).
\end{array}
\end{align*}
\]
This works because $g_2 \cdot (g_1 \cdot x) = (g_2 g_1) \cdot x$. ($g \cdot x =$ action of $g$ on $x$).

Example: The group $\mathbb{IR}$ of reals acts on $S^1 = \{z \in \mathbb{C} | |z| = 1\} = \{e^{i \theta} | \theta \in \mathbb{R}\}$:

$x \cdot z = e^{ixz}$ complex multiplication. It's an action since $(x + y) \cdot z = e^{i(x+y)z} = e^{ix} e^{iy} z$.

We get an action groupoid $\mathbb{IR} \times S^1 \Rightarrow S^1$.

We will show that $\mathbb{IR} \times S^1 \Rightarrow S^1$ is isomorphic to the fundamental groupoid $\Pi S^1$ of $S^1$.

Note first that for any $(x, z) \in \mathbb{IR} \times S^1$ we have a path $\gamma_{x,z} : [0, x] \to S^1$, $\gamma_{x,z}(t) = e^{it}$ for $t \in [0, x]$. Hence we get a morphism $e^{ixz} : \gamma_{x,z} \to \gamma_{x+y,z}$ in $\Pi S^1$ (see Remark 11 above).

For $z \in S^1$, $x, y \in \mathbb{IR}$, $\gamma_{y, e^{ix} z} \circ \gamma_{x,z} = \gamma_{x+y, e^{iy} z}$.

This gives us a functor $\mathbb{IR} \times S^1 \to \Pi S^1$, $(e^{ix} \leftarrow z, e^{iy} \leftarrow z) \mapsto e^{i(x+y)} z$.

While "\(\gamma\)" is bijective on objects, it's not at all clear that it's fully faithful.

On the other hand if there is a cover $\{U_1, U_2\}$ of $S^1$ and functors $\tau_1 : \Pi U_1 \to \mathbb{IR} \times S^1$ $\tau_2 : \Pi U_2 \to \mathbb{IR} \times S^1$ so that $\Pi U_1 \xrightarrow{\tau_1} \mathbb{IR} \times S^1 \xleftarrow{\tau_2} \Pi U_2$ is a pushout of $\Pi U_1 \to \Pi U_1 \cup \Pi U_2 \to \Pi U_2$ then we're done by Lemma 33.1 since pushouts are unique up to an isomorphism.

So let $U_1 = S^1 \setminus \{1\}$ $U_2 = S^1$, and $\tau_1 : U_1 \to S^1 \setminus \{1\}$.

We'll show next time that this choice works.