Last time, we defined natural isomorphisms and observed that if \( \text{if } G \xrightarrow{\beta} D \) and \( D \) is a groupoid, then \( \alpha \) is a natural isomorphism.

- Defined what it means for two categories to be equivalent. Main reason we care:
  - If \( \psi \colon X \to Y \) is a homotopy equivalence, then \( \text{if } \psi \colon \Pi X \to \Pi Y \) is part of an equivalence of categories.

Proved: If \( F \colon C \to D \) is part of an equivalence of categories then \( \psi \colon \text{Hom}_C(c, c') \to \text{Hom}_D(Fc, Fc') \) is a bijection if \( c, c' \in C \) and \( \psi \) is a natural isomorphism of categories.

Consequence: If \( \psi \colon X \to Y \) is a homotopy equivalence then \( \psi \) is an isomorphism of groups.

Conversely,

**Lemma 3.12** Let \( F \colon C \to D \) be a fully faithful and essentially surjective functor. Then \( F \) is part of an equivalence of categories: \( F \) is a functor \( G \colon D \to C \) and natural isomorphisms \( \alpha \cdot \text{id}_C \to GF, \beta \cdot F \Rightarrow \text{id}_D \).

**Proof** Since \( F \colon C \to D \) is essentially surjective, for every object \( d \in D \) we can choose \( c \in C \) (\( c \) depends on \( d \)) and an iso \( \beta_d : F(c) \to d \). Define \( G : D \to C \) on objects by setting \( G(b) \) to be that \( c \). Then \( \beta_d : F(G(d)) \overset{=} \to d \).

We need to define \( G \) on morphisms. So let \( d \to d' \) be a morphism. Since \( \beta_d, \beta_d' \) are iso's \( \exists! \) morphism \( \psi : FG(d) \to FG(d') \) so that the diagram

\[
\begin{array}{ccc}
F_G(d) & \overset{\beta_d}{\to} & d \\
\down{\psi} & & \down{\psi} \\
FG(d') & \overset{\beta_d'}{\to} & d'
\end{array}
\]

Since \( F \) is fully faithful, \( \exists! \psi : G(d) \to G(d') \) so that \( F\psi = \psi \).

We define \( G(d) = \psi \). This defines \( G \) on morphisms. It's not hard to show that \( G(\text{id}_d) = \text{id}_{G(d)} \) and that \( G \) preserves composition, hence \( G \) is a functor.

Since for any morphism \( \alpha : d \to d' \) the diagram

\[
\begin{array}{ccc}
FG(d) & \overset{\beta_d}{\to} & d \\
\down{FG(\alpha)} & & \down{FG(\alpha)} \\
FG(d') & \overset{\beta_d'}{\to} & d'
\end{array}
\]

commutes and...
since \( \beta_d \)'s are isomorphisms, \( \beta = (\beta_d)_{d \in D} \) in a natural isomorphism
\( \beta: FG \to \id_D \).

It remains to construct a natural isomorphism \( \alpha: \id G \to GF \). For any \( c \in G \) we

an isomorphism \( \beta_c: FG(c) \to F(c) \). Since \( F \) is fully faithful, \( \exists! \alpha_c: c \to GF(c) \)

so that \( \alpha_c \circ \beta_c = (\beta_c \circ \beta)^{-1} \). Remains to check: \( \forall \ c, c' \in G \) the diagram

\[
\begin{array}{ccc}
GF(c) & \xrightarrow{\alpha_c} & c \\
\downarrow \GF(c) & & \downarrow \GF(c) \\
GF(c') & \xrightarrow{\alpha_{c'}} & c'
\end{array}
\]

commutes. We know that \( \GF(c) \circ (\beta_c \circ \beta)^{-1} = F(c) \)

so that \( \alpha_c \circ \beta_c = (\beta_c \circ \beta)^{-1} \).

Since \( F \) is faithful,
\( F(\alpha_c \circ \alpha_c) = F(\beta_c \circ \beta_c) \circ \alpha_c \circ \beta_c = GF(c) \circ \alpha_c \)

and we are done.

So far we have defined fundamental groups and fundamental groups, but we
don't really know how to compute them except for a few easy cases.

For example, if \( X \subseteq \mathbb{R}^n \) is convex then \( \pi_1 X \simeq \text{Pair}(X) \) and \( \pi_1 (\ldots) \times \{e\} \), the trivial

Note: If \( X \subseteq \mathbb{R}^n \) is convex then \( X \) is homotopy equivalent to a 1 point space \( \{x\} \). Why?

We can also easily show: if \( X \) is homotopy equivalent to \( \{x\} \), then \( \pi_1 (X, x_0) \simeq \{e\} \) for any \( x_0 \in X \). Why? But we don't know how to compute \( \pi_1 (S^k) \), the

fundamental group of a circle. For this we need a theorem. We'll use

Brown–Seifert–van Kampen. To state this theorem, we need a definition.

Definition (pushout). Let \( C \) be a category, \( a, b, c \) three objects and \( f_1: c \to a, f_2: c \to b \) two

morphisms in \( C \): \( a \xleftarrow{f_1} c \xrightarrow{f_2} b \). A pushout of \( a \xleftarrow{f_1} c \xrightarrow{f_2} b \) is an object \( p \) of \( C \)


and two morphisms $\iota_1 : a \to p$, $\iota_2 : b \to p$ so that (1) \[ \begin{array}{ccc}
\iota_1 & \downarrow & \iota_2 \\
b & \rightarrow & p
\end{array} \] commutes and (ii) A object $d$ of $C$ and any pair of morphisms $k_1 : a \to d$, $k_2 : b \to d$

so that \[ \begin{array}{ccc}
k_1 & \\b & \rightarrow & p
\end{array} \] commutes \[ \exists ! k : p \to d \] so that

\[ \begin{array}{ccc}
k_1 & \downarrow & k_2 \\
b & \rightarrow & p
\end{array} \]

commutes: \( k \circ i_1 = k_1, k \circ i_2 = k_2 \)

---

**Example.** Let $C = \text{Set}$, the category of sets. Given three sets and two functions $A \xrightarrow{f_1} B$, $A \xrightarrow{f_2} C$ we can construct the corresponding pushout as follows. Consider the disjoint union $A \cup B$ with its canonical inclusions $j_1 : A \to A \cup B$, $j_2 : B \to A \cup B$.

Let $\sim$ be the smallest equivalence relation so that $j_1(f_1(x)) \sim j_2(f_2(x)) \quad \forall x \in C$.

Let $D = (A \cup B)/\sim$, and let $q : A \cup B \to (A \cup B)/\sim = P$ denote the quotient map.

Define $i_1 : A \to P$ to be $i_1 = q \circ j_1$, Similarly $i_2 = q \circ j_2$.

Then, $\forall x \in C$,

\[
(i_1 \circ f_1)(x) = (q \circ j_1 \circ f_1)(x) = q(f_1(f_1(x))) = q(f_2(f_2(x))) = (i_2 \circ f_2)(x)
\]

so

\[ \begin{array}{ccc}
h_1 & \\b & \rightarrow & p
\end{array} \]

commutes.

Given any set $D$ and any pair of functions $k_1 : A \to D$, $k_2 : B \to D$, we have $\tilde{k} : A \cup B \to D$ defined by

\[ \tilde{k}(y) = \begin{cases} k_1(a) & \text{if } y = j_1(a) \\ k_2(b) & \text{if } y = j_2(b) \end{cases} \]

Now suppose $k_1(f_1(x)) = k_2(f_2(x)) \quad \forall x \in C$.

Then $\tilde{k} (j_1(f_1(x))) = k_1(f_1(x))$ and $\tilde{k} (j_2(f_2(x))) = k_2(f_2(x))$.

If $\tilde{k}$ is constant on the equivalence classes of $\sim$ hence gives rise to a well-defined map $k : (A \cup B)/\sim \to D$ so that $k \circ q = \tilde{k}$.

Moreover $k \circ i_1 = k \circ q \circ j_1 = k \circ j_1 = k_1$ and similarly $k \circ i_2 = k_2$.

Moreover $k \circ i_1 = k \circ q \circ j_1 = k \circ j_1 = k_1$ and similarly $k \circ i_2 = k_2$.
It remains to check uniqueness of \( k \). So suppose \( b : P \to D \) is another function so that \( l_1 \circ k_1 = k \), \( l_2 \circ k_2 = k_2 \). Then
\[
\log \circ d_1 \circ k_1, \quad \log \circ d_2 \circ k_2.
\]
By the universal property of \( A \xleftarrow{d_1} A \cup B \xrightarrow{d_2} B \)
\[
\log \circ \widehat{\lambda}, \quad \text{(Informally } \log \big| A \xrightarrow{k_1} A \cup B \xrightarrow{k_2} B \Rightarrow \log \circ \widehat{\lambda} \).
\]
\[
\Rightarrow \log \circ \lambda \circ \omega. \quad \text{Since } \lambda \text{ is onto, } \lambda = \widehat{\lambda}.
\]

**Example.** Let \( C = \text{Top}, \) \( A, B, C \) are three spaces and \( f_1 : C \to A, f_2 : C \to B \) are two continuous maps. Then the pushout of \( A \xleftarrow{f_1} C \xrightarrow{f_2} B \) exists and its construction is almost the same as in Set: we consider \( A \cup B \) with the coproduct topology ( \( U \subseteq A \cup B \) is open \( \iff (A \cap U \) is open \( \cap A, \) \( B \cup U \) is open \( \cap B) \)) \( f_1 : A \to A \cup B, f_2 : B \to A \cup B \) are continuous. We again take \( \sim \) to be the smallest equivalence relation so that \( f_1(\lambda(x)) \sim f_2(\lambda(x)) \) \( \forall \lambda \subseteq C \) and give
\[
P = \left( A \cup B \right) / \sim \quad \text{the quotient topology. Then } \left( l_1 = q \circ f_1 : A \to P \right)
\]
and \( l_2 = q \circ f_2 : B \to P \) are continuous. Then \( f_2 \quad \text{and} \quad f_1 \)
commutes in \( \text{Top} \) and has the desired \( f_2 \quad \text{universal property.} \)

**Remark.** If \( b \xleftarrow{\sim} a \) and \( b \xleftarrow{\sim} a' \) are two different pushouts in \( E \)
of \( b \xleftarrow{f} c \xrightarrow{g} a \), then \( p \) and \( p' \) are isomorphic (why?). So pushouts are unique only up to an isomorphism.

**Example.** Let \( X \) be a space, \( \{ U_1, U_2 \} \) an open cover of \( X \). Then
\[
U_1 \xleftarrow{f_1} X \xleftarrow{f_2} U_2
\]
is a pushout of \( U_1 \xleftarrow{f_1} U_1 \cap U_2 \xrightarrow{f_2} U_2 \) where \( f_1, f_2 \)
are the inclusions. This because given a space \( Y \) and two continuous maps
\( k_1 : U_1 - Y, \ k_2 : U_2 - Y \) so that \( k_1 f_1 = k_1 f_2 \), \( k_1 \mid U_1 \cap U_2 = k_2 \mid U_1 \cap U_2 \). Hence \( k_1, k_2 \)
define a unique continuous map \( k : X - Y \) so that \( k \mid U_1 = k_1, \ k \mid U_2 = k_2. \)