Last time: constructed a functor $\Pi: \text{Top} \to \text{Groupoid}$, $X \mapsto \Pi X = \text{fund groupoid of } X$
- defined natural transformations $\xi: \xi_0 \to \xi_1$, $c \mapsto \xi_c: F(c) \to G(c)$
so that $\xi \circ c = c'$ in $\mathcal{C}$ the diagram

$$
\begin{array}{ccc}
F(c) & \xrightarrow{\xi_c} & G(c) \\
F(f) \downarrow & & \downarrow G(f) \\
F(c') & \xrightarrow{\xi_{c'}} & G(c')
\end{array}
$$

(\text{Theorem 30.5})

Proved: if $\psi \circ F: X \to Y$ are two homotopic maps then $F$ gives rise to a natural transformation $\Pi F: \Pi X \Rightarrow \Pi Y$. (ie $\Pi X \xrightarrow{\Pi \psi} \Pi Y$).

Definition A natural transformation $\xi: \xi_0 \to \xi_1$ is a natural isomorphism if $\psi \circ \xi_0 \equiv \xi_1 \circ \psi$ for $\xi_0: F(c) \to G(c)$ is an isomorphism in $\mathcal{D}$.

Remark If $\mathcal{D}$ is a groupoid, then any morphism in $\mathcal{D}$ is an isomorphism by definition. So for any category $\mathcal{C}$, any two functors $F, G: \mathcal{C} \to \mathcal{D}$ any natural transformation $\alpha: F \Rightarrow G$ is a natural isomorphism.

Back to fundamental groupoids

Suppose $X$ and $Y$ are two homotopy equivalent spaces. Then $\exists$ continuous maps $\psi: X \to Y$, $\varphi: Y \to X$ and homotopies $\psi \circ \varphi \simeq_{\xi} \text{id}_X$, $\varphi \circ \psi \simeq_{G} \text{id}_Y$. (1)

By Theorem 30.5 and the above remark, we have natural isomorphisms

$$
\Pi F: \Pi (\psi \circ \varphi) \Rightarrow \Pi (\varphi \circ \psi) \Rightarrow \Pi \text{id}_Y
$$

Since $\Pi$ is a functor $\Pi (\psi \circ \varphi) \Rightarrow \Pi \psi \circ \Pi \varphi$, $\Pi (\varphi \circ \psi) \Rightarrow \Pi \varphi \circ \Pi \psi$.

Hence (1) $\Rightarrow$

$$
\Pi \psi \circ \Pi \varphi \simeq_{\Pi F} \text{id}_{\Pi X}, \quad \Pi \varphi \circ \Pi \psi \simeq_{\Pi G} \text{id}_{\Pi Y}
$$

Definition Two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if $\exists$ functors $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\alpha: GF \Rightarrow \text{id}_\mathcal{C}$, $\beta: FG \Rightarrow \text{id}_\mathcal{D}$.

Example If $X, Y$ are two homotopy equivalent spaces then their fundamental groupoids $\Pi X, \Pi Y$ are equivalent categories.
To have more examples of equivalent categories (and equivalent groupoids) and to understand the implications of such an equivalence we need to develop some tools. First, a remark and some definitions.

**Remark.** If \( \xi : D \xrightarrow{\text{Id}} D \) is a natural isomorphism, then \( x \times x' \in D \), \( \alpha^{-1} : F(x) \to G(x) \) is invertible in \( D \). \( \Rightarrow \) we get a family of isomorphisms \( \{ \alpha^{-1}_x : \xi \Rightarrow \text{id}_{F(x)} \}_{x \in \mathcal{C}_0} \). It's easy to check that \( x \times x' \in D \), \( \xi \Rightarrow \text{id}_{F(x')} \) commutes. Hence

\[
\begin{align*}
\alpha^{-1}_x & \quad \alpha^{-1}_{x'} \quad \xi \Rightarrow \text{id}_{F(x')} \\
G(x) & \quad \uparrow \quad \uparrow (\xi^{-1})_x \\
F(x) & \quad \downarrow \quad \downarrow (\xi^{-1})_{x'} \\
G(x') & \quad \Rightarrow \quad F(x')
\end{align*}
\]

we have a natural isomorphism \( \alpha^{-1} : G \Rightarrow F \). And, in fact, "being naturally isomorphic" is an equivalence relation on the collection of all functors between any two fixed categories.

**Definition.** Let \( F : \mathcal{C} \to \mathcal{D} \) be a functor.

- \( F \) is **faithful** if \( \forall c, c' \in \mathcal{C}, \; F : \text{Hom}_\mathcal{C}(c, c') \to \text{Hom}_\mathcal{D}(F(c), F(c')) \) is injective.
- \( F \) is **full** if \( \forall c, c' \in \mathcal{C}, \; F : \text{Hom}_\mathcal{C}(c, c') \to \text{Hom}_\mathcal{D}(F(c), F(c')) \) is surjective.
- \( F \) is **fully faithful** if \( F \) is full and faithful.

**Examples.**

1. The functor \( U : \text{Top} \to \text{Set} \), \( U : (X, \tau) \mapsto (Y, \tau) \), \( U(f) = f \) is faithful but not full: there may be functions \( f : X \to Y \) that are not continuous.

2. Recall that there is a category \( \mathbb{1} \) with one object \( * \) and one morphism \( \text{id}_* \). For any category \( \mathcal{C} \), there is a functor \( G : \mathcal{C} \to \mathbb{1} \), \( G(\{c \mapsto c'\}) = \text{Id}_* \), for all \( c \in \mathcal{C} \).

- \( G : \mathcal{C} \to \mathbb{1} \) is full, but if \( \exists c, c' \in \mathcal{C} \) s.t. \( |\text{Hom}_\mathcal{C}(c, c')| > 1 \), \( G \) is not faithful.

3. For any category \( \mathcal{E} \) and any object \( c \in \mathcal{E} \), we have the monoid \( \text{Hom}_\mathcal{E}(c, c) \), which we can consider as a one object category \( \pi_1(\mathcal{E}, c) \). The one object \( \pi_1(\mathcal{E}, c) \in \mathcal{C} \) and \( \text{Hom}_{\pi_1(\mathcal{E}, c)}(c, c) = \text{Hom}_{\mathcal{E}}(c, c) \).

Then the "inclusion functor" \( i : \pi_1(\mathcal{E}, c) \to \mathcal{E} \), \( i(c \xrightarrow{x} c') = (c \xrightarrow{x} c') \) is fully faithful.
Definition. A functor $F: C \to D$ is essentially surjective if any object $d$ of $D$ is isomorphic to $F(c)$ for some object $c \in C$.

Example. Let $X$ be a path connected space. Then for any point $x_0 \in X$ the inclusion $\pi_1(X, x_0) \to \pi_1 X$ is essentially surjective: $\forall x \in X$ there is a path $\gamma$ from $x_0$ to $x$. Hence $\pi_1 X$ is an isomorphism in $\pi_1 X$ from $x_0 = \gamma(x_0)$ to $x$.

Lemma 31.1. Let $F: C \to D$ be (part of) an equivalence of categories (so there is a functor $G: D \to C$ and natural isomorphisms $\alpha : d \to GF, \beta: F \circ G = Id_D$). Then $F$ is full, faithful and essentially surjective.

Proof. We first prove that $F$ (and $G$) are faithful. For any morphism $c \to c'$ in $C$ the diagram

$\begin{array}{ccc}
c & \xleftarrow{c} & GF(c) \\
\downarrow & \alpha_c & \downarrow GF(\alpha) \\
c' & \xleftarrow{\alpha'_c} & GF(c')
\end{array}$

commutes.

Suppose $\gamma, \gamma' : Hom_c(c, c')$ and $F(\gamma) = F(\gamma')$. Then $GF(\gamma) = GF(\gamma')$. Since $\beta$ commutes for $\gamma, \gamma'$ and since $\alpha_c, \alpha_c'$ are isos,

$\gamma = (\alpha_c')^{-1} \circ GF(\gamma) \circ \alpha_c = (\alpha_c')^{-1} \circ GF(\gamma') \circ \alpha_c = \gamma'$.

$\Rightarrow$ $F$ is faithful. By the same argument $G$ is faithful.

To show that $F$ is full, given $\mu : F(c) \to F(c')$ we need to find $\gamma : c \to c'$ with $F(\gamma) = \mu$.

Let $\gamma = (\alpha_{c'})^{-1} \circ G(\mu) \circ \alpha_c$ so that $\begin{array}{ccc}
c & \xleftarrow{\alpha_c} & GF(c) \\
\downarrow & \circ \downarrow & \circ \downarrow GF(\mu) \\
c' & \xleftarrow{\alpha'_c} & GF(c')
\end{array}$

commutes. Since $\alpha_c, \alpha_c'$ are isos it follows that $GF(\gamma) = G(\mu)$.

Since $G$ is faithful, $F(\gamma) = \mu$.

Finally we argue that $F$ is essentially surjective. For any object $d \in D$,

$\beta_d : F(G(d)) \to d$ is an isomorphism. So $\forall d \in D, c \in G(d) \in C$ is an object so that $F(c)$ is isomorphic to $d$. $\Rightarrow$ $F$ is essentially surjective.

$\Box$
"Application" Suppose \( f : X \to Y \) is a homotopy equivalence. Then for any point \( x_0 \in X \), the groups \( \pi_1(X, x_0) \) and \( \pi_1(Y, f(x_0)) \) are isomorphic.

**Reason** \( f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0)) \) is an equivalence of categories by Theorem 30.5.

By Lemma 31.1, the functor \( f_* \) is fully faithful. Hence \( \forall x_0 \in X \)

\[
\forall x_0 : \text{Hom}_\pi(x_0, x_0) \to \text{Hom}_\pi(f(x_0), f(x_0)) \text{ is a bijection. But}
\]

(i) \( \text{Hom}_\pi(x_0, x_0) = \pi_1(X, x_0) \)
(ii) \( \text{Hom}_\pi(f(x_0), f(x_0)) = \pi_1(Y, f(x_0)) \) and

(iii) \( f_* : \text{Hom}_\pi(x_0, x_0) \to \text{Hom}_\pi(f(x_0), f(x_0)) \) preserves composition.

Hence \( f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0)) \) is an isomorphism of groups.

**Lemma 31.1** has a useful converse:

**Lemma 31.2** Let \( F : C \to D \) be a fully faithful and essentially surjective functor. Then \( F \) is part of an equivalence of categories: \( \exists \) a functor \( G : D \to C \) and natural isomorphisms \( \alpha : \text{id}_C \to G \circ F, \beta : F \circ G \to \text{id}_D \).

We'll prove it next time...