Last time, we constructed the fundamental groupoid $\Pi X$ of a space $X$: objects of $\Pi X$ are points of $X$, morphisms are homotopy classes of paths.

- Observed: in any category $\mathcal{C}$, for any object $c$ of $\mathcal{C}$, $\text{Hom}_\mathcal{C}(c, c)$ is a monoid.

  If $\mathcal{C}$ is a groupoid then $\text{Hom}_\mathcal{C}(c, c)$ is a group.

  If $\mathcal{C} = \Pi X$, $x \in X$, $\text{Hom}_\Pi X(x, x) = \pi_1(X, x)$ the fundamental group of $X$ (at $x$).

  Elements of $\pi_1(X, x)$ are homotopy classes of loops - paths that start and end at $x$.

- Introduced the pair groupoid $\text{Pair}(X)$ of a set $X$ and proved:

  If $X \subseteq \mathbb{R}^n$ is a convex set then $\Pi X \cong \text{Pair}(X)$ is an isomorphism of groupoids.

### Proposition 30.1

Let $X, Y$ be two spaces and $\varphi : X \to Y$ a continuous map. Then $\varphi$ gives rise to a functor $\varphi_* : \Pi X \to \Pi Y$ given a morphism $x \xrightarrow{[f]} x'$ in $\Pi X$, $\varphi_* ([f]) = \varphi (x) \xrightarrow{\varphi (f)} \varphi (x')$.

#### Proof

We need to check:

(i) $\varphi_*$ is well-defined

(ii) $\varphi_*$ preserves composition and identity morphisms.

(i) Suppose $x \xrightarrow{f_0} x'$, $x \xrightarrow{f_1} x''$ are two paths in $X$ and $f_0 \simeq_f f_1$ rel $\{0, 1\}$ (so $[f_0] = [f_1]$ in $\Pi X$). Then $\varphi_0 : \{0, 1\} \to Y$ is a homotopy rel $\{0, 1\}$ from $\varphi_0 f_0$ to $\varphi_0 f_1$. Thus $\varphi_0 f_0 \simeq \varphi_0 f_1$ rel $\{0, 1\}$, $[\varphi_0 f_0] = [\varphi_0 f_1]$. Hence $\varphi_* : (\Pi X)_\Delta \to (\Pi Y)_\Delta$, $([f]) \mapsto [\varphi_* f]$ is well-defined.

(ii) If $c_x : \{0, 1\} \to X$ is the constant path at $x$, then $\forall s \in \{0, 1\}$,

$$\varphi (c_x)_s = \varphi (c_x(s)) = \varphi (c_x(1)) = \varphi (c_x(0)) = \varphi (c_x)$$

Thus $\varphi_0 c_x = c_{\varphi (x)}$ and $\varphi_* (\text{id}_x) = \varphi_* (c_x) = [\varphi (c_x)] = \text{id}_{\varphi (x)}$ for $x \in X$.

If $x'' \xleftarrow{\theta} x' \xrightarrow{\varphi} x$ are two composable paths in $X$ then $\varphi (x'') \xrightarrow{\varphi \circ \theta} \varphi (x') \xrightarrow{\varphi (f)} \varphi (x)$ are two composable paths in $Y$. And
Theorem 30.2 There is a functor $\Pi$ from the category $\text{Top}$ of topological spaces and continuous maps to the category $\text{Groupoid}$ of groupoids and functors. It is given by $\Pi (X, Y) = \Pi X \times_{\Pi Y} \Pi Y$ for all spaces $X, Y$ and all continuous maps $\varphi : X \to Y$.

Proof. We need to check that $\Pi$ preserves identity morphisms and compositions. Suppose $X$ is a space, $\text{id}_X : X \to X$, $\text{id}_X(x) = x$ the identity map. Then

$$\forall x \xrightarrow{\text{id}_X} x' \in \Pi X,$$

$$\Pi(\text{id}_X)(x) \xrightarrow{\text{id}_X} \Pi X, \quad \Pi(\text{id}_X)(x) \xrightarrow{\text{id}_X} \Pi X$$

$$= \text{id}_X(x) \xrightarrow{\text{id}_X} x.$$

If $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$ is a pair of continuous maps then $\forall x \xrightarrow{\varphi} x' \in \Pi X$

$$\Pi(\varphi)(x) \xrightarrow{\Pi(\psi)} \Pi Y \xrightarrow{\Pi(\varphi)} \Pi Z$$

$$= \Pi(\varphi \circ \psi)(x) \xrightarrow{\Pi(\psi \circ \varphi)} \Pi Y \xrightarrow{\Pi(\varphi \circ \psi)} \Pi Z.$$

Corollary 30.3 If $X$ and $Y$ are two homeomorphic spaces then $\Pi X$ and $\Pi Y$ are isomorphic groupoids.

Corollary 30.3 follows from the following category-theoretic lemma.

Lemma 30.4 Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between two categories. Suppose $c \xrightarrow{c'}$ is an isomorphism in $\mathcal{C}$. Then $F(c) \xrightarrow{F(c')} F(d)$ is an isomorphism in $\mathcal{D}$.

Proof. Since $\varphi$ is an isomorphism, $F(c) \xrightarrow{c'} F(c) \text{ s.t. } F(\varphi) = \text{id}_{F(c)}$, $\varphi \circ \varphi = \text{id}_{F(c)}$. Then $F(\varphi) \circ F(\varphi) = F(\varphi \circ \varphi) = F(\text{id}_{F(c)}) = \text{id}_{F(c)}$.

Similarly, $F(\varphi) \circ F(\varphi) = \text{id}_{F(c)}$. Therefore, $F(\varphi)$ is an isomorphism in $\mathcal{D}$. 

\[ (\varphi \cdot g) \cdot (\varphi \cdot f) = \varphi \cdot (g \cdot f) \\text{ s.t.} \\ 0 \leq s \leq \frac{1}{2} \]
Recall: two spaces may be homotopy equivalent without being homeomorphic. We aim to show: if \( X \) and \( Y \) are homotopy equivalent then their fundamental groupoids are “equivalent” in some sense. To say “equivalent” about groupoids we need an analogue of homotopy for functors.

**Definition** Let \( F, G : \mathcal{C} \to \mathcal{D} \) be two functors. A natural transformation \( \alpha : F \Rightarrow G \) (also written as \( \phi : \mathcal{C}_0 \to \mathcal{D}_1 \) that assigns to each object \( c \) of \( \mathcal{C} \) a morphism \( F(c) \to G(c) \) of \( \mathcal{D} \) so that \( \forall \) morphism \( c \to c' \) in \( \mathcal{C} \) the diagram \( \xymatrix{ F(c) \ar[r]^{\alpha_c} & G(c) } \) commuted in \( \mathcal{D} \), i.e.

\[
\xymatrix{ F(c) \ar[r]^{\alpha_c} \ar[d]_{\alpha_{F(c)}} & G(c) \ar[d]^{\alpha_{G(c)}} \quad G(f) \circ \alpha_c = \alpha_{G(c)} \circ F(f) }
\]

**Theorem 30.5** Suppose \( \psi \simeq \psi : X \to Y \) are two homotopic maps between spaces. Then the homotopy \( F : \times [0,1] \to Y \) defines a natural transformation \( \Pi F : \Pi Y \Rightarrow \Pi Y \).

**Remark** We’ll use 30.5 to prove: if \( \psi : X \to Y \) is a homotopy equivalence then then \( \Pi \psi \equiv \psi_* : \Pi_1(X,x_0) \to \Pi_1(Y,y_0) \) is an isomorphism of groups.

To prove Theorem 30.5 we need a technical lemma.

**Lemma 30.6** Let \( Y \) be a space, \( H : [0,1] \times [0,1] \to Y \) a continuous map. Define paths

\[
\mu, \nu, \tau : [0,1] \to Y \\
\mu(t) = F(0,t), \quad \nu(t) = F(t,0), \\
\tau(s) = F(s,1) \quad \text{and} \quad \sigma(t) = F(1,t).
\]

Then \( \mu \times \nu \simeq \sigma \times \tau \) rel \( \{0,1\} \times \{0,1\} \), hence \( [\mu, \nu] = [\tau, \sigma] \) as morphisms in \( \Pi Y \).

**Proof** Note that the closed disk \( D = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\} \) is homeomorphic to the square \( \{0,1\}^2 \).
We also have a homotopy \( \mu: \mathbb{R}^2 \rightarrow Y \) from the lower to the upper half of the circle. The composite map \( H \circ \hat{\circ} \circ \circ : [0,1]^2 \rightarrow Y \) is a desired homotopy.

**Proof of 30.5** For any point \( x \in X \) (an object of \( \Pi X \)) we need \( (\Pi F)_\psi : \psi(x) \rightarrow \psi(x') \), a morphism in \( \Pi Y \) so that for any morphism \( x \xrightarrow{f} x' \) in \( \Pi X \) (i.e. for the homotopy class of a path \( x \xrightarrow{\bar{f}} x' \)) the diagram

\[
\begin{array}{ccc}
\psi(x) & \xrightarrow{\Pi Y (f)} & \psi(x') \\
\Pi F_x & \uparrow & \Pi F_{x'} \\
\end{array}
\]

commutes in \( \Pi Y \), i.e.

\[
\Pi Y (f) \circ \Pi F_x = \Pi F_{x'} \circ \Pi f.
\]

(\( \circ \) = composition in \( \Pi Y \)).

Consider \( F_x(t) = F(x, t), \ t \in [0,1] \). \( F_x \) is a path in \( Y \) from \( F(x,0) = \psi(x) \) to \( F(x,1) = \psi(x') \). Define \( \Pi F_x := [ F_x ] \). Similarly we have \( F_x(t) = \psi(x,t) \).

Now consider \( H : [0,1]^2 \rightarrow Y \), \( H(s,t) := F( f(s), t ) \).

Then \( H(s,0) = F(f(s),0) = \psi(f(s)) \), \( H(s,1) = F(f(s),1) = \psi(f(s)) \)

\( H(0,t) = F(x,t) = F_x(t), \ H(1,t) = F(x',t) = F_{x'}(t) \).

Now apply Lemma 30.6.