Last time: Finished proving: a Hausdorff manifold is paracompact ⇒ its connected components are 2nd countable.

- Defined a homotopy between two continuous maps and homotopy classes of maps.
- Defined categories and sketched a proof that \( h\text{Top} \) (topological spaces + homotopy classes of continuous maps) is a category.

**Example (of a category)** Let \( G \) be a group. Then \( G \) defines a category \( BG : BG \) has only one object, call it \(*\). \( \text{Hom}_{BG}(*,*) = G \). The "composition"

\[
\text{Hom}_{BG}(*,*) \times \text{Hom}_{BG}(*,*) \to \text{Hom}_{BG}(*,*)
\]

is the group multiplication.

\( \text{id}_* = e \), the identity element in \( G \).

**Definition** Let \( \mathcal{C} \) be a category. A morphism \( f : c \to d \) in \( \mathcal{C} \) is an isomorphism if \( \exists g : d \to c \) so that \( fg = \text{id}_d \), \( gf = \text{id}_c \).

**Example** If \( \mathcal{C} = \text{Set} \), then an isomorphism \( f : X \to Y \) is an invertible map.

If \( \mathcal{C} = \text{Top} \), the category of topological spaces and continuous maps then \( f : X \to Y \) is an isomorphism in \( \text{Top} \) ⇒ \( f \) is a homeomorphism.

**Definition** A continuous map \( f : X \to Y \) between two topological spaces is a homotopy equivalence ⇒ \( \exists g : Y \to X \), continuous, and two homotopies \( gf \simeq \text{id}_X \), \( fg \simeq \text{id}_Y \).

**Remark** \( g \) is called a homotopy inverse of \( f \).

**Example** \( X = \{ * \} \) 1 point space, \( Y = \mathbb{R}^n \), \( f : \{ * \} \to \mathbb{R}^n \) is a homotopy equivalence.

Consider \( g : \mathbb{R}^n \to \{ * \} \), \( g(x) = * \) \( \forall x \in \mathbb{R}^n \). Then \( g(f(x)) = g(0) = * \) ⇒ \( gf = \text{id}_{\{ * \}} \).

\( (f \circ g)(x) = f(g(x)) = 0 \) \( \forall x \in \mathbb{R}^n \) and \( F : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n \), \( F(x,t) = tx \) is a homotopy from \( f \circ g \) to \( \text{id}_{\mathbb{R}^n} \).

**Interpretation** Note that in \( h\text{Top} \), *space* \( X \), the identity morphism \( X \to X \) is the homotopy
Consequently, if \( f: X \to Y \) is a continuous map, then \( g: Y \to X \) is a homotopy inverse of \( f \Rightarrow [f, g] \) and \([g, f] = [\text{id}_Y, \text{id}_X] \).

Consequently

\[ f: X \to Y \text{ is a homotopy equivalence } \iff [f, f]: X \to Y \text{ is an isomorphism in the category } h\text{Top}. \]

**Definition.** Two topological spaces \( X \) and \( Y \) are homotopy equivalent if they are isomorphic in \( h\text{Top} \), i.e., \( \exists \) a homotopy equivalence \( f: X \to Y \), i.e., \( \exists \) an isomorphism \( [f]: X \to Y \) in \( h\text{Top} \).

**Important fact.** The circle \( S^1 \) is not homotopy equivalent to a 1-point space \( \ast \).

This fact is hard to prove without some tools.

We'll prove it by constructing for every space \( X \) a category \( \Pi X \), called the fundamental groupoid. We'll prove: (i) a homotopy equivalence \( f: X \to Y \) defines an equivalence \( \Pi f: \Pi X \to \Pi Y \) of fundamental groupoids.

1. \( \Pi \ast = 1 \), a category with exactly one object \( \ast \) and 1 morphism \( \text{id}_\ast: \ast \to \ast \).
2. \( \Pi S^1 \) is not equivalent to \( 1 \).

We start by defining functors.

**Definition.** Let \( C, D \) be two categories. A functor \( F: C \to D \) is a pair of maps

\[ F_0: C_0 \to D_0 \text{ on objects, } F_1: C_1 \to D_1 \text{ on morphisms that are suitably compatible} \]

and, additionally, \( F_1 \) preserves the composition and units. More precisely we require that

1. A morphism \( c \to c' \) in \( C \), \( F_0(c) \) is a morphism in \( D \) from \( F_0(c) \) to \( F_0(c') \).
2. A morphism \( F_1(f) \circ F_1(g) = F_1(f \circ g) \).
3. A pair of morphisms \( x \to y \to z \) is a morphism in \( C \) which are composable

\[ F_1(f \circ g) = F_1(f) \circ F_1(g) \]

**Notation.** We'll drop the indices 0 and 1 and write \( F \) for both \( F_0 \) and \( F_1 \).
We have a functor $+ : (LCH, \text{proper maps}) \to (\text{compact Hausdorff}, \text{continuous maps})$:

$$X \xrightarrow{f} Y \mapsto x^+ = x \cup_0 x_1 \xrightarrow{f^+} Y \cup_0 Y_s = Y^+$$

There is a functor $U : \text{Top} \to \text{Set}$ that forgets topology:

$$U((X, T_X) \xrightarrow{f} (Y, T_Y)) = X \xrightarrow{f} Y$$

There is a functor $\text{ind} = \text{indiscrete topology}$ from $\text{Set}$ to $\text{Top}$:

$$\text{ind}(X, \tau_X) = (X, 1_\emptyset \times \tau_f) \xrightarrow{\tau_f} (Y, 1_\emptyset \times \tau_f)$$

Definition: A category $\mathcal{C}$ is a groupoid if every morphism in $\mathcal{C}$ is an isomorphism (i.e., invertible).

$\mathcal{C}_0 = \text{sets}, \mathcal{C}_1 = \text{bijections}$ is a groupoid.

If $G$ is a group, $B G$ with $(B G)_0 = \emptyset$, $(B G)_1 = G$ is a groupoid.

Let $G$ and $H$ be two groups. Their disjoint union is not a group but it is a groupoid:

$\mathcal{C} = B G \sqcup B H : \mathcal{C}_0 = \{x, y\}$.

$\text{Home}(x, x) = G$, $\text{Home}(y, y) = H$, $\text{Home}(x, y) = \emptyset$, $\text{Home}(y, x) = \emptyset$.

Back to topology

Definition (relative homotopy): Let $W, X$ be two spaces, $A \subseteq W$ a subspace and $f_0, f_1 : W \to X$ two continuous maps with $f_0|_A = f_1|_A$.

$f_0$ is homotopic to $f_1$ relative to $A$ (notation: $f_0 \simeq f_1 \text{ rel } A$) if $\exists F : W \times [0, 1] \to X$, continuous, with $F(w, 0) = f_0(w)$, $F(w, 1) = f_1(w)$ for $w \notin W$ (i.e., $F$ is a homotopy from $f_0$ to $f_1$) and $F(a, t) = f_0(a) = f_1(a)$ for all $(a, t) \in A \times [0, 1]$, i.e., $F$ leaves the points of $A$ fixed.

We write $f_0 \simeq f_1 \text{ rel } A$. 
Special case. \( W = \{0, 1\}, \quad A = \{0, 1\}. \) Then \( f_0, f_1 : W \to X \) are paths with \( f_0(0) = f_1(0), \)
\( f_0(1) = f_1(1). \) 
\( f_0 \simeq f_1 \text{ rel } \{0, 1\} \iff \) the homotopy \( F \) fixes the endpoints:

\[
\begin{array}{c}
\text{x} \\
\xymatrix{
& y \\
\text{x} \ar[r]_{f_0} & \text{y} \\
& x \\
\text{y} \ar[u]_{f_1} \\
\end{array}
\]

Exercise. \( f \simeq g \text{ rel } \{0, 1\} \) is an equivalence relation on the set of paths in a space \( X. \)

**Definition** (the fundamental groupoid). Let \( X \) be a topological space. The fundamental groupoid \( \Pi X \) of \( X \) is defined as follows:

- objects \( (\Pi X)_0 = X \), the set of points in \( X \)
- morphisms: \( \text{Hom}_{\Pi X}(x, y) = \) homotopy classes of paths from \( x \) to \( y \) rel. end points

\[
= \{ \gamma : [0, 1] \to X \mid \gamma(0) = x, \gamma(1) = y \} / \sim \quad \text{where} \quad \gamma_1 \sim \gamma_2 \iff \gamma_1 \simeq \gamma_2 \text{ rel } \{0, 1\}
\]

Composition: concatenation of (homotopy classes) of paths.

Recall: given \( \sigma \xrightarrow{s} y \xleftarrow{r} x \), \( \sigma \ast \gamma(s) \):=

\[
\left\{ \begin{array}{l}
\gamma(2s) \quad 0 \leq s \leq 1/2 \\
\gamma(2s-1) \quad 1/2 \leq s \leq 1
\end{array} \right.
\]

To show that the purported category \( \Pi X \) has a well-defined composition we need

**Lemma 28.1** Let \( X \) be a space, \( \sigma, \sigma' : [0, 1] \to X \), \( \gamma, \gamma' : [0, 1] \to X \) four paths with
\( \sigma \simeq \gamma \text{ rel } \{0, 1\}, \) \( \sigma \simeq \sigma' \text{ rel } \{0, 1\} \) and \( \sigma(0) = \sigma'(0) = \gamma(0) = \gamma'(0). \) Then
\( \sigma \ast \gamma \simeq \sigma' \ast \gamma' \text{ rel } \{0, 1\}. \)

We'll prove the lemma next time and finish construction/definition of the fundamental groupoid \( \Pi X. \)