Last time: locally compact Hausdorff + $\sigma$-compact $\Rightarrow$ paracompact.

- (stated without proof) closed subspaces of paracompact spaces are paracompact.
- Paracompact spaces are normal

- If $\{A_\alpha\}_{\alpha \in \Lambda}$ is locally finite, then $\bigcup_{\alpha \in \Lambda} A_\alpha = \bigcup_{\alpha \in \Lambda} \overline{A_\alpha}$.

Goals for today:
1) If $X$ is paracompact then $\forall$ open cover $\exists$ a partition of 1 subordinate to the cover.

2) A manifold $M$ is paracompact $\iff M$ is a disjoint union of 2nd countable Hausdorff manifolds.

**Lemma 26.1 (the shrinking lemma)** Suppose $X$ is paracompact, $\{U_\alpha\}_{\alpha \in A}$ an open cover. Then $\exists$ locally finite open cover $\{V_\alpha\}_{\alpha \in A}$ with $\overline{V_\alpha} \subseteq U_\alpha$ $\forall \alpha$ [where $V_\alpha = \emptyset$ is allowed].

**Proof** Since paracompact spaces are regular, $\forall U_\alpha, \forall x \in U_\alpha$ $\exists$ open nbd $O$ of $x$ with $\overline{O} \subseteq U_\alpha$. We get an open cover $\mathcal{O}$ of $X$:

$\mathcal{O} = \{ O \subseteq X \text{ open} \mid \overline{O} \subseteq U_\alpha \text{ for some } \alpha \in A \}$.

Since $X$ is paracompact, $\mathcal{O}$ has a locally finite open refinement $\mathcal{W}$. For each $W \in \mathcal{W}$ choose $\alpha \in A$ st $\overline{W} \subseteq U_\alpha$, that is, choose a function $f: W \to A$ with $\overline{W} \subseteq U_{f(W)}$.

Now for each $\alpha \in A$ let $V_\alpha = \bigcup_{W \in \mathcal{W} \atop f(W) = \alpha} \overline{W}$. (if $\{ W \mid f(W) = \alpha \} = \emptyset$, set $V_\alpha = \emptyset$).

Since $\mathcal{W}$ is locally finite, $\overline{V_\alpha} = \bigcup_{W \in \mathcal{W} \atop f(W) = \alpha} \overline{W} = \bigcup_{f(W) = \alpha} \overline{W}$; and since $\overline{W} \subseteq U_{f(W)}$ for all $W$ with $f(W) = \alpha$, we have $\overline{V_\alpha} \subseteq U_\alpha$.

Remains to check: the cover $\{ V_\alpha \}_{\alpha \in A}$ is locally finite.

Choose $x \in X$. Since $\overline{V_\alpha}$ is locally finite, $\exists$ a nbd $N$ of $x$ so that $N \cap V_\alpha = \emptyset$ for all but finitely many $W \in \mathcal{W}$.

$\Rightarrow A' = \{ f(W) \mid N \cap W \neq \emptyset \}$ is finite.

Since $V_\alpha \cap N \neq \emptyset$ only for $\alpha \in A'$, the cover $\{ V_\alpha \}_{\alpha \in A}$ is locally finite. $\square$
Theorem 26.2 Let $X$ be a paracompact space, $(U_a)_{a \in A}$ an open cover of $X$. Then

$\exists$ a partition of $1$ $\{p_a : X \to \bigcup_{a \in A} U_a \text{ with } \text{supp } p_a \subseteq U_a \}.$

Proof. By the shrinking lemma $\exists$ a locally finite open cover $(V_a)_{a \in A}$ with $\overline{V_a} \subseteq U_a$, $\forall a$.

Applying the shrinking lemma again we get a locally finite open cover $(W_a)_{a \in A}$ with $W_a \subseteq V_a$, $\forall a$. Since $\overline{W_a}$, $X \setminus V_a$ are closed and disjoint, Urysohn's lemma $\Rightarrow$

$\exists$ continuous function $\varphi : X \to \{0, 1\}$ so that $\varphi_a | \overline{V_a} \equiv 1$ and $\varphi_a | X \setminus V_a \equiv 0$.

Since $\{ V_a \}_{a \in A}$ is locally finite, $\{ \overline{V_a} \}_{a \in A}$ is also locally finite (exercise).

Since $\varphi_a | X \setminus V_a \equiv 0$, $\text{supp } \varphi_a \subseteq V_a$. Since $\{ \overline{V_a} \}_{a \in A}$ is locally finite, $\{ \text{supp } \varphi_a \}_{a \in A}$ is also locally finite. $\Rightarrow$ $\varphi(x) = \sum_{a \in A} \varphi_a(x)$ is a well-defined continuous function.

Since $\bigcup_{a \in A} W_a = X$ and $\varphi_a | W_a \equiv 1$ and $\varphi_a(x) \geq 0 \forall x$, $\varphi(x) > 0 \forall x$.

Define $f_a(x) = \varphi_a(x) / \varphi(x)$.

Then $\text{supp } f_a(x) = \text{supp } \varphi_a(x)$ and $\sum_{a \in A} f_a(x) = \frac{1}{\varphi(x)} \sum_{a \in A} \varphi_a(x) = 1$.

$\therefore \{ f_a \}_{a \in A}$ is a desired partition of $1$.

Theorem 26.3 A manifold $M$ is paracompact $\iff$ $M$ is a disjoint union of Hausdorff and countable manifolds.

Proof $(\Rightarrow)$ Any manifold is locally compact. Any 2nd countable locally compact Hausdorff space is $\sigma$-compact (exercise). Hence by 25.1 any Hausdorff 2nd countable manifold is paracompact. Finally any disjoint union of paracompact spaces is paracompact.

We prove $(\Leftarrow)$ by proving two propositions and using the fact that any space is the disjoint union of its connected components.

Proposition 26.4 A connected locally compact paracompact space is $\sigma$-compact.

Proposition 26.5 If $X$ is $\sigma$-compact and every point of $X$ has a 2nd countable nbd, then $X$ is 2nd countable.
Proof of 29.5. Since $X$ is $\sigma$-compact, $X = \bigcup_{i=1}^{\infty} C_i$, where $C_i$ are compact.

For each $x \in X$ find a 2nd countable nbd $N_x$.

= each $C_i$ has an open cover by 2nd countable open sets, hence by finitely many open 2nd countable sets ($C_i$ is compact). Take their union. We get a 2nd countable open set $U_i$ with $C_i \subseteq U_i$.

Then $X = \bigcup_{i=1}^{\infty} C_i \subseteq \bigcup_{i=1}^{\infty} U_i \subseteq X \Rightarrow X = \bigcup_{i=1}^{\infty} U_i$. Since a countable union of 2nd countable open sets is 2nd countable, $X$ is 2nd countable.

We'll prove 26.4 next time.