Last time: (X/G) is Hausdorff if G is an LCH group acting properly on an LCH space X.

- notion of connectedness: [0, 1] is connected.
- X is connected if there is no discrete space Y and any continuous function f: X → Y, f is constant.

Proposition 23.1 Images of connected spaces are connected: if X is connected and f: X → Z is continuous then f(X) ⊂ Z is connected (in subspace topology).

Proof: Suppose Y is discrete, g: f(X) → Y continuous. Then g ∘ f: X → Z is continuous. Since X is connected, g ∘ f is constant. ⇒ g ∘ f(X) → Z is connected.

Since Y and g are arbitrary, f(X) is connected.

Proposition 23.2 Let \{X_i\}_{i ∈ I} be a collection of connected subspaces of a space Z with \(X_i ∩ X_j = \emptyset\) for all \(i ≠ j\). Then \(\bigcup_{i ∈ I} X_i\) is connected.

Proof: Suppose Y is discrete, \(q: \bigcup_{i ∈ I} X_i → Y\) continuous. Since \(Y\), \(X_i\) is connected, \(q|_{X_i}\) is constant; let \(y_i = q(X_i)\). We now argue that \(y_i = y_j\) ∀ i, j: since \(X_i ∩ X_j = \emptyset\), \(∃ x_j ∈ X_i \cap X_j\). Then \(y_i = q(x_j) = y_j\). ⇒ \(g\) is constant. ⇒ \(\bigcup_{i ∈ I} X_i\) is connected.

Corollary: \(R\) is connected.

Proof: For any \([a, b] ⊂ R\), \(φ: [0, 1] → [a, b], \ φ(t) = t + b(1-t)a\) is a homeomorphism. ⇒ ∀ \(a < b\), \([a, b]\) is connected.

\(R = \bigcup_{n ∈ Z} [-n, n]\), which is connected by 23.2.

Note: ∀ \(a < b\), \([a, b]\) is homeomorphic to \(R\), hence is connected.

Proposition 23.3 The closure of a connected subspace is connected. More generally, if \(A \subset X\) is connected and \(A \subset E \subset \overline{A}\), then E is connected.

Proof: Suppose \(V, W \subset E\), \(V \cap W = \emptyset\), \(V \cup W = E\) and \(V, W\) are closed in \(E\).
To prove that E is connected it's enough to show: \( V = \emptyset \) or \( W = \emptyset \).  
(Why?)

Since \( V, W \) are closed in \( E \), \( \exists \tilde{V}, \tilde{W} \subset X \) closed with \( V = E \cap \tilde{V}, W = E \cap \tilde{W} \).

Then \( A \cap V = A \cap (E \cap \tilde{V}) = A \cap \tilde{V} \) (since \( A \subset E \)) and
\[
A \cap W = A \cap (E \cap \tilde{W}) = A \cap \tilde{W}.
\]

\( \Rightarrow \) \( A \cap V, A \cap W \) are closed in \( A \). Additionally,
\[
(A \cap V) \cap (A \cap W) \leq V \cap W = \emptyset,
\]
and \( A = A \cap E = A \cap (V \cup W) = (A \cap V) \cup (A \cap W) \).

Since \( A \) is connected, either \( V \cap A = \emptyset \) or \( W \cap A = \emptyset \). It's no loss of generality to assume \( V \cap W = \emptyset \). Then \( A = A \cap V = A \cap \tilde{V} \).

\( \Rightarrow \) \( A \subset \tilde{V} \). Since \( \tilde{V} \) is closed, \( \overline{A} \subset \tilde{V} \). Since \( E \subset \overline{A} \), \( E \subset \tilde{V} \). \( \Rightarrow E = E \cap \tilde{V} \).

Then \( W = E \cap \tilde{W} = (E \cap \tilde{V}) \cap \tilde{W} = (E \cap \tilde{V}) \cap (E \cap \tilde{W}) = V \cap W = \emptyset \), and we're done. \( \square \)

\[\begin{align*}
\text{Consider } A &= \{ (x,y) \in \mathbb{R}^2 \mid x > 0, y = \sin \frac{1}{x} \}. \quad \text{Since } f : (0,\infty) \rightarrow A, f(x) = (x, \sin \frac{1}{x}) \\
&\text{is a continuous surjective map and since } (0,\infty) \approx \mathbb{R} \text{ is connected, } A \text{ is connected.}
\end{align*}\]

By 23.3, \( A \) is connected. Note that the set of limit points of \( A \) is \( A' = \{ 0 \} \times [0,1] \).

\[\Rightarrow \overline{A} = \left( \{ 0 \} \times [0,1] \right) \cup A \text{ is connected.}\]

**Connected components**

Let \( X \) be a space.

Define a relation \( \sim \) on \( X \) by \( x \sim y \iff \exists A \subset X \) connected with \( x, y \in A \).

Claim \( \sim \) is an equivalence relation.

Check. \( x \sim x \) since we may take \( A = \{ x \} \).

- \( x \sim y \Rightarrow y \sim x \), clearly.
- \( x \sim y \) and \( y \sim z \) \( \exists A, B \) connected with \( x, y \in A, y, z \in B \). Then \( A \cap B \neq \emptyset \) and so, since \( A, B \) are connected, \( A \cup B \) is connected. Since \( x, z \in A \cup B, x \sim z \).

**Definition** A connected component of a space \( X \) is an equivalence class of the
Proposition 23.4 1) Connected components of a space \( X \) are connected and closed in \( X \).
2) If \( Y \subseteq X \) is connected, then \( Y \subseteq A \) is a connected component of \( X \).

Proof: If \( Y \subseteq X \) is connected, then \( \forall x,y \in Y, x \neq y \Rightarrow Y \subseteq \text{connected component of } X \).

1) Let \( C \) be a connected component of \( x \in X \). Then \( C = \bigcup_{A \subseteq X} \). By 23.2, \( C \) is connected.

By 23.3, \( \bar{C} \) is connected. Since \( x \in \bar{C} \), \( \bar{C} \) is one of the \( A \)’s in the definition of \( C \).

\( \Rightarrow \bar{C} \subseteq C \). \( \Rightarrow C = \bar{C} \) and so is closed.

Remark Consider \( A \subseteq \mathbb{R} \) with subspace topology. \( \forall a,b \in A \) with \( a < b \) \( \exists r \in \mathbb{R} \setminus A \) \( \forall a < r < b \Rightarrow A = ((-\infty, r) \cup (r, +\infty)) \). \( \Rightarrow a,b \) are in different connected components of \( A \). \( \Rightarrow \) The connected components of \( A \) are singletons; they are not open.

Lemma 23.5 (Intermediate value theorem). Let \( X \) be a connected space, \( f: X \to \mathbb{R} \) continuous, \( a,b \in X \) with \( f(a) < f(b) \). Then \( \exists c \in (f(a), f(b)) \exists x \in X \) s.t. \( c = f(x) \).

Proof: Suppose not. \( \exists c \notin f(X) \Rightarrow X = f^{-1}([c]) = f^{-1}((-\infty, c) \cup (c, +\infty)) = f^{-1}((-\infty, c)) \cup f^{-1}((c, +\infty)) \). \( f^{-1}((-\infty, c)) \) \( f^{-1}((c, +\infty)) \) are disjoint, open and nonempty, since \( \forall a \in f^{-1}((-\infty, c)) \), \( b \in f^{-1}(c, +\infty) \). This contradicts connectedness of \( X \).

Path connectedness

Definition A path in a space \( X \) from \( x \in X \) to \( y \in X \) is a continuous map \( \gamma: [0,1] \to X \) with \( \gamma(0) = x \), \( \gamma(1) = y \).

We say in this case: “\( x \) is connected to \( y \) by a path \( \gamma \).”

Definition A space \( X \) is path connected if any two points in \( X \) can be connected by
Lemma 23.6. If $X$ is path connected, Then $X$ is connected.

Proof. Fix $x_0 \in X$. Since $X$ is path connected, $\forall y \in X$ there is a path $\gamma : [0,1] \to X$ from $x_0$ to $y$ (i.e. $\gamma(0) = x_0$, $\gamma(1) = y$). Since $\gamma([0,1]) \subseteq X$ is connected, $y$ lies in the connected component of $X$ containing $x_0$. $\Rightarrow X$ has only one connected component. $\Rightarrow X$ is connected.

Lemma 23.7. Let $A = \{(x, \sin(1/x)) \in \mathbb{R}^2 \mid x \in (0, \infty)\}$. Then $A$ is connected but not path connected.

Proof. Suppose $A$ is path connected. Then $\exists$ a path $\gamma : [0,1] \to A$ s.t. $\gamma(0) = (0,0)$ and $\gamma(1) \notin A$. $A \setminus A = \{0\} \times [-1,1]$ is closed in $\mathbb{R}^2$, hence closed in $A$. $\Rightarrow \gamma'(A \setminus A)$ is closed in $[0,1]$ (and contains 0). Let $d = \sup \gamma'(A \setminus A)$. Then, since $\gamma'(A \setminus A)$ is closed, $d < \gamma'(A \setminus A)$.

$\Rightarrow \gamma'(d,1) \subseteq A$. $\forall t = (x(t), y(t))$ with $x, y : [0,1] \to \mathbb{R}$ continuous.

For $t > d$, $(x(t), y(t)) \in A \Rightarrow y(t) = \sin \left(\frac{1}{x(t)}\right)$ for $t > d$.

$y(d) \notin A \setminus A = \{0\} \times [-1,1] \Rightarrow x(d) = 0$. Since $x(t) \to x(d)$ as $t \to d$, $\exists$ a sequence $t_n \in [d,1]$ s.t. $t_n \to d$ and $x(t_n) = \frac{1}{\pi/2 + 2\pi n}$. Then $y(t_n) = \sin(\frac{\pi}{2} + 2\pi n) = (-1)^{n+1}$

On the one hand, since $y$ is continuous, $y(t) \to y(d)$ as $t \to d$.

On the other hand, $y(t_n) = (-1)^{n+1}$ doesn’t converge. Contradiction.

$\therefore A$ is not path connected.