Last time, \( X \) has one-point Hausdorff compactification \( X^+ \Leftrightarrow X \) in LCH and noncompact.

- A point compactifications of LCH spaces are unique (up to a unique homeomorphism)
- \( \text{f} : X \to Y \) is proper if \( \forall C \subseteq Y \text{ compact}, \text{f}^{-1}(C) \text{ is compact}. \)
- In general, if \( \text{f} : X \to Y \) is continuous, \( \text{f}^+: X^+ \to Y^+ \) need not be continuous.

**Theorem 21.1** A continuous map \( \text{f} : X \to Y \) between two LCH spaces extends to a continuous map \( \text{f}^+: X^+ \to Y^+ \Leftrightarrow \text{f} : X \to Y \) is proper.

**Proof** We tacitly assume \( Y \subseteq Y^+, X \subseteq X^+, \ Y^+ = Y \cup \{ \infty \}, \ X^+ = X \cup \{ \infty \} \).

\( \Rightarrow \) Suppose \( \text{f}^+: X^+ \to Y^+ \) is continuous. Let \( C \subseteq Y \) be compact. Then \( Y^+ \setminus C \) is an open nbh of \( \infty \), and in particular, is open. \( \Rightarrow C \) is closed in \( Y^+ \to (\text{f}^+)^{-1}(C) \) is closed in \( X^+ \to (\text{f}^+)^{-1}(C) \cap \{ \infty \} = \emptyset \) since \( \text{f}^+(\infty) = \infty \notin C. \)

\( \Rightarrow (\text{f}^+)^{-1}(C) = \text{f}^{-1}(C) \to \text{f}^{-1}(C) \) is closed in \( X \to \text{f}^{-1}(C) \) is compact in \( X^+ \), hence compact in \( X \). \( \Rightarrow \) is proper.

\( \Leftarrow \) Suppose \( \text{f} \) is proper. We need to show: \( \forall U \subseteq Y^+ \text{ open}, \ (\text{f}^+)^{-1}(U) \) is open in \( X^+ \).

Case 1 \( U \subseteq Y^+ \setminus \{ \infty \} = Y \). Then \( (\text{f}^+)^{-1}(U) = \text{f}^{-1}(U) \) is open in \( X \) (since \( \text{f} \) is continuous) \( \Rightarrow (\text{f}^+)^{-1}(U) \) is open in \( X \).

Case 2 \( \infty \notin U \). Then \( Y^+ \setminus U \) is a compact subset of \( Y \).

\( \Rightarrow (\text{f}^+)^{-1}(Y^+ \setminus U) = \text{f}^{-1}(Y^+ \setminus U) \) is compact in \( X \) since \( \text{f} \) is proper.

\( \Rightarrow (\text{f}^+)^{-1}(U) = X^+ \setminus (\text{f}^+)^{-1}(Y^+ \setminus U) \) is open in \( X^+ \).

\( \square \)

**Corollary 21.2** A proper continuous map between LCH spaces is closed.

**Proof** Suppose \( \text{f} : X \to Y \) is proper and \( C \subseteq X \) is closed. Then \( \text{f}^+: X^+ \to Y^+ \) is continuous, \( \tilde{C} := C \cup \{ \infty \} \) is closed in \( X^+ \) (since \( X^+ \setminus \tilde{C} = X \setminus C \) is open in \( X^+ \)).

Since \( X^+ \) is compact, \( \tilde{C} \) is compact (closed subsets of compact spaces are compact).

Since \( \text{f}^+ \) is continuous, \( \text{f}^+(\tilde{C}) = \text{f}(C) \cup \{ \text{f}(\infty) \} \) is compact in \( Y^+ \).

Since \( Y^+ \) is Hausdorff, \( \text{f}(C) \cup \{ \text{f}(\infty) \} \) is closed in \( Y^+ \).

\( \Rightarrow \text{f}(C) = (\text{f}(C) \cup \{ \text{f}(\infty) \}) \cap \tilde{Y} \) is closed in \( Y \).

\( \square \)
Our next goal: quotients of LCH spaces by proper actions of LCH topological groups are Hausdorff. First some definitions.

Def. Recall that a group $G$ is a set $G$ with a distinguished element $e \in G$
and two operations $\mu: G \times G \to G$, $\mu(a, b) = ab$ and $\text{inv}: G \to G$, $\text{inv}(g) = g^{-1}$
so that i) $eg = g = ge$, ii) $g \in G$, ii) $\mu$ is associative: $(ab)c = a(bc)$, $\forall a, b, c \in G$ and
iii) $gg^{-1} = e = g^{-1}g$, $\forall g \in G$.

A topological group is a topological space $G$ with a distinguished
element $e \in G$ and continuous maps $\mu: G \times G \to G$, $\text{inv}: G \to G$ so that
(i) $eg = g = ge$, (ii) $\mu$ is associative and (iii) $\text{inv}(g) = \text{inv}(g)g = e$.

Examples. Any group with a discrete topology.

• $\mathbb{R}^n$ w. standard topology, $\mu(x, y) = x + y$, $\text{inv}(x) = -x$, $e = 0$.

• $\text{GL}(n, \mathbb{R}) = \{ A \in \mathbb{R}^{n \times n} \mid \det A \neq 0 \}$ w. subspace topology, $e = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,
$\mu(A, B) = AB$, $\text{inv}(A) = A^{-1}$.

$C^\times = \{ z \in \mathbb{C} \mid z \neq 0, e = 1, \mu(z, w) = zw, \text{inv}(z) = z^{-1} \}$.

Definition. A (left) action of a top. group $G$ on a top. space $X$ is a continuous map
$G \times X \to X$, $(g, x) \mapsto g \cdot x$
So that i) $e \cdot x = x$, $\forall x \in X$
2) $(g_1g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$, $\forall g_1, g_2 \in G$, $\forall x \in X$.

Ex. Any top. group $G$ acts on $G$ by left multiplication: $g \cdot x = gx$, $\forall g, x \in G$.

Ex. $\text{GL}(n, \mathbb{R})$ acts on $\mathbb{R}^n$ by $A \cdot x = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.

Ex. $C^\times$ acts on $\mathbb{C}^n$ by $\lambda \cdot (\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}) = (\begin{pmatrix} \lambda w_1 \\ \vdots \\ \lambda w_n \end{pmatrix})$, multiplication of a vector by a nonzero scalar.
Definition Let $G \times X \to X$ be an action. The orbit of $x \in X$ is the subspace $G \cdot x := \{ g \cdot x \mid g \in G \}$.

Consider $GL(n, \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n$. $GL(n, \mathbb{R}) \cdot \delta = \{ A \delta \mid A \in GL(n, \mathbb{R}) \} = \delta$.

If $\bar{z} \in \mathbb{R}^n$ and $\bar{x} \neq \bar{y}$, then $\forall \bar{y} \in \mathbb{R}^n \exists A \in GL(n, \mathbb{R})$ s.t. $A \bar{y} = \bar{y}$.

$GL(n, \mathbb{R}) \cdot \bar{z} = \mathbb{R}^n \setminus \{ \bar{y} \}$.

Proposition 21.3 Suppose a top. group $G$ acts on a space $X$. Then $x \sim x' \iff \exists g \in G$ s.t. $g \cdot x = x'$ is an equivalence relation. The equivalence classes of $\sim$ are exactly the $G$-orbits.

Proof exercise.

Definition An action of a top. group $G$ on a space $X$ is proper if the map $a : G \times X \to X \times X$, $a(g, x) = (x, g \cdot x)$ is proper.

Example An action of a group $G$ on itself by left multiplication is proper. This is because $a : G \times G \to G \times G$, $a(g, x) = (x, g \cdot x)$ has a continuous inverse: $a^{-1}((x, y)) = (x, y^{-1})$, and so $a$ is a homeomorphism, hence proper.
Lemma 21.4 An action of a compact group on a Hausdorff space is proper.

Proof Let $X$ be Hausdorff, $G$ a compact group, $G \times X \rightarrow X$ an action.

We need to show: \( \forall C \subseteq X \times X \text{ compact}, \quad a^{-1}(C) \subseteq G \times X \text{ is compact} \), where, as above

\[ a : G \times X \rightarrow X \times X \quad \text{where} \quad a(g,x) = (x,g.x) \]

Note that since $\pi : X \times X \rightarrow X$, $\pi_1(x_1,x_2) = x_1$ is continuous, $\pi_1(C)$ is compact $\Rightarrow G \times \pi_1(C)$ is compact. Now, $a^{-1}(C) = \{(g,x) \mid (x,g.x) \in C \} \subseteq G \times \pi_1(C)$.

Since $C$ is compact and $X \times X$ is Hausdorff, $C$ is closed in $X \times X$. Since $a$ is continuous, $a^{-1}(C)$ is closed. Since $a^{-1}(C) \subseteq G \times \pi_1(C)$, $a^{-1}(C)$ is compact.

Thus $a$ is proper, and the action of $G$ on $X$ is proper by definition. \( \square \)