Last time: Defined metrics and metric spaces

- Define a topology \( T \) on a set \( X \): \( T \subseteq \mathcal{P}(X) \) so that

  i) \( \emptyset, X \in T \) (i.e., \( \emptyset, X \) are open)

  ii) if \( U, V \in T \) then \( U \cap V \in T \) (intersections of two open sets are open)

  iii) if \( \{ U_x \}_{x \in A} \subseteq T \) then \( \bigcup_{x \in A} U_x \in T \) (arbitrary unions of open sets are open)

- We saw: a metric \( d \) on a set \( X \) gives rise to a topology \( T_d \) on \( X \).

- two different metrics may give rise to the same topology

- there are topologies that come from no metrics.

**Definition** Let \( (X, T) \) be a topological space. A subset \( C \subseteq X \) is closed iff \( X \setminus C \) is open, i.e., \( X \setminus C \in T \).

**Exercise** Let \( (X, T) \) be a topological space. Then

i) \( \emptyset, X \) are closed

ii) If \( C_1, C_2 \subseteq X \) are closed then so is \( C_1 \cup C_2 \).

iii) If \( \{ C_x \}_{x \in A} \) is a family of closed sets then \( \bigcap_{x \in A} C_x \) is closed.

**Example** Consider \( \mathbb{R} \) with the standard topology. Then for \( a, b \in \mathbb{R}, a < b \), the closed interval \([a, b]\) is closed. (Why?) Note: \( \bigcup_{n=1}^{\infty} \left[ \frac{1}{n}, 1 - \frac{1}{n} \right] = (0, 1) \) which is not closed. So arbitrary unions of closed sets need not be closed.

Similarly, \( \bigcap_{n=1}^{\infty} \left( -\frac{1}{n}, \frac{1}{n} \right) = [0, 1] \), which is not open.

**Remark** Being open and closed is not mutually exclusive: if \( (X, T) \) is a topological space, then \( X, \emptyset \) are both open and closed.

If \( T = \mathcal{P}(X) \) then any \( A \subseteq X \) is opened and closed.

**Continuity**

Recall: a function \( f: \mathbb{R} \to \mathbb{R} \) is continuous at \( x_0 \in \mathbb{R} \) if \( \forall \varepsilon > 0 \exists \delta > 0 \) so that
A function $f: \mathbb{R} \to \mathbb{R}$ is continuous if it is continuous at every $x_0 \in \mathbb{R}$.

This $\varepsilon-\delta$ definition of continuity generalizes to arbitrary metric spaces:

**Definition 2.1** Let $(X, d_X)$ and $(Y, d_Y)$ be two metric spaces. A function $f: X \to Y$ is continuous at $x_0 \in X$ if $\forall \varepsilon > 0 \exists \delta > 0$ so that

$$d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon.$$  

A function $f: X \to Y$ is continuous if it is continuous at every $x_0 \in X$.

Recall: in a metric space $(X, d)$, $B_r(x) = \{ y \in X \mid d(x, y) < r \}$.

**Lemma 2.2** Equation (2.1) is equivalent to:

$$B_\varepsilon(x_0) \subseteq f^{-1}(B_\varepsilon(f(x_0))).$$

**Proof** (2.1) holds

iff $x \in B_\delta(x_0) \Rightarrow f(x) \in B_\varepsilon(f(x_0))$

iff $B_\delta(x_0) \subseteq f^{-1}(B_\varepsilon(f(x_0))).$  

**Lemma 2.3** Let $(X, d_X)$, $(Y, d_Y)$ be two metric spaces. A function $f: X \to Y$ is continuous in the sense of Definition 2.1 $\iff$

$\forall U \subseteq Y$ open (w.r.t. $d_Y$) the preimage $f^{-1}(U)$ is open in $X$ (w.r.t. $d_X$).

**Proof** ($\Rightarrow$)

Suppose $U \subseteq Y$ is open. If $f^{-1}(U) = \emptyset$, it's open. Suppose $f^{-1}(U) \neq \emptyset$. Pick any $x_0 \in f^{-1}(U)$. Then $f(x_0) \in U$. Since $U$ is open, $\exists \varepsilon > 0$ so that $B_\varepsilon(f(x_0)) \subseteq U$. Since $f$ is continuous at $x_0$, $\exists \delta > 0$ so that $d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon$.

By Lemma 2.2, $B_\varepsilon(x_0) \subseteq f^{-1}(B_\varepsilon(f(x_0)))$. Since $f^{-1}(B_\varepsilon(f(x_0))) \subseteq f^{-1}(U)$,

$B_\varepsilon(x_0) \subseteq f^{-1}(U)$. Since $x_0 \in f^{-1}(U)$ is arbitrary, $f^{-1}(U)$ is open in $X$.

($\Leftarrow$) Suppose $\forall U \subseteq Y$ open, $f^{-1}(U)$ is open, $x_0 \in X$, $\varepsilon > 0$. 

We know: open balls are open. \( \Rightarrow B_x(f(x_0)) \subseteq Y \) is open. Hence, by assumption
\[ f^{-1}(B_x(f(x_0))) \] is open in \( X \). Since \( f(x_0) \in B_x(f(x_0)) \),
\( x_0 \in f^{-1}(B_x(f(x_0))) \). Since \( f^{-1}(B_x(f(x_0))) \) is open \( \exists \delta > 0 \)
so that \( B_x(x_0) \subseteq f^{-1}(B_x(f(x_0))) \). By Lemma 2.2
\[ d_x(x,x_0) < \delta \Rightarrow d_Y(f(x),f(x_0)) < \varepsilon, \quad \text{i.e.} \ f \text{ is continuous} \]
at \( x_0 \) in the sense of Definition 2.1. Since \( x_0 \) is arbitrary, \( f \) is continuous. \( \square \)

We now turn Lemma 2.3 into a definition:

**Definition** Let \( (X,T_X), (Y,T_Y) \) be two topological spaces. A function \( f: X \to Y \)
is continuous if \( Y \) open subset \( U \in T_Y \), the preimage \( f^{-1}(U) \) is open, i.e.,
\[ f^{-1}(U) \in T_X. \]

(Non) example Consider a step function \( f: \mathbb{R} \to \mathbb{R}, \ f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \)
\( f \) is not continuous, since
\[ f^{-1}((-\frac{1}{2},\frac{1}{2})) = [0,0) \]
which is not open in \( \mathbb{R} \).

**Comparing topologies** Let \( X \) be a set, \( T_1, T_2 \) two topologies on \( X \). We say that
\( T_1 \) is smaller than \( T_2 \) (and \( T_2 \) is bigger) iff \( T_1 \subseteq T_2 \).

Other terminology: smaller = coarser = weaker
bigger = finer = stronger.

Note For any set \( X \), i.e., \( X \) is the smallest possible topology on \( X \), while \( P(X) \), the set
of all subsets of \( X \) is the largest possible topology.

**Exercise** For any topological space \( (Y,T_Y) \) and any set \( X \), any function \( f: (X,T_X) \to (Y,T_Y) \)
is continuous. Similarly, any function \( h: (X,P(X)) \to (Y,T_Y) \) is continuous.
Subspace topology

**Lemma 24** Let \((X, T_X)\) be a topological space, \(Y \subseteq X\) a subset. The set
\[ T^Y = \{ U \subseteq Y \mid \exists \tilde{U} \in T_X \text{ so that } U = \tilde{U} \cap Y \} \]
is a topology on \(Y\).
Moreover \(T^Y\) is the smallest topology on \(Y\) so that the inclusion
\[ i : Y \hookrightarrow X, \quad i(Y) = Y \]
is continuous.

**Proof**

\(i\) \(Y = X \cap Y \in T^Y \) and \(\emptyset = \emptyset \cap Y \in T^Y \).

\(ii\) If \(U, V \in T^Y\) then \(\exists \tilde{U}, \tilde{V} \subseteq X\) open (i.e. \(\tilde{U}, \tilde{V} \in T_X\)) so that \(U = \tilde{U} \cap Y, V = \tilde{V} \cap Y\). Then \(U \cap V = (\tilde{U} \cap Y) \cap (\tilde{V} \cap Y) = (\tilde{U} \cap \tilde{V}) \cap Y\). Since \(T_X\) is a topology, \(\tilde{U} \cap \tilde{V} \in T_X\).
\[ \Rightarrow U \cap V = (\tilde{U} \cap \tilde{V}) \cap Y \in T^Y. \]

\(iii\) Similarly suppose \(\{ U_a \}_{a \in A} \subseteq T^Y\). Then \(\exists \tilde{U}_a \subseteq T_X\) s.t. \(U_a = \tilde{U}_a \cap Y\).
\[ \Rightarrow \bigcup_{a \in A} U_a = \bigcup_{a \in A} (\tilde{U}_a \cap Y) = (\bigcup_{a \in A} \tilde{U}_a) \cap Y \subseteq T^Y \quad \text{since } U \tilde{U}_a \in T_X. \]

If \(T'\) is a topology on \(Y\) and \(i : (Y, T') \rightarrow (X, T_X)\) is continuous, then \(\forall \tilde{U} \in T_X\)
\[ i^{-1}(\tilde{U}) = \tilde{U} \cap Y \subseteq T'. \] Hence \(T^Y \subseteq T'\).

**Bases**

**Definition** Let \((X, T)\) be a topological space. A subset \(B \subseteq T\) is a basis for \(T\) if any \(U \in T\) is a union of elements of \(B\).

**Example (Silly)** \(B = T\) is a basis for \(T\).

**Example** Let \((X, d)\) be a metric space, \(T_d\) metric topology: \(U \in T_d\)
\[ \Leftrightarrow \forall x \in U \exists r > 0 \text{ s.t. } B_r(x) \subseteq U. \] Consequently \(U = \bigcup_{x \in U} B_r(x)\).
Hence \(B = \{ B_r(w) \mid x \in X, r > 0 \}\) is a basis for \(T_d\).