Last time: 2nd countable + regular ⇒ metrizable.

Lindelöf spaces; 2nd countable ⇒ Lindelöf.

Tietze extension theorem: \( X \) be a normal topological space, \( F \subset X \) a closed subset; \( f : F \to [0,1] \) a continuous function. Then \( \exists \) a continuous function \( \tilde{f} : X \to [0,1] \), an extension of \( f \), so that \( \tilde{f} |_F = f \).

Promised to show: Moore/Nemytskiǐ’s plane is not normal.

Sketch of proof: Recall: the plane is \( X := \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\} \) with a larger than standard topology. In particular, every point in \( L := \{(x,y) \in X \mid y = 0\} \) (with the subspace topology) is open (and closed) and \( L \subset X \) is closed.

Any function \( f : L \to [0,1] \) is continuous.

Notation: For any two topological space \( W \) and \( Z \)

\[ C^0(W, Z) \equiv C(W, Z) = \text{set of continuous functions from } W \text{ to } Z. \]

So \( C^0(L, [0,1]) \equiv [0,1]^R = \text{all functions from } \mathbb{R} \text{ to } [0,1]. \)

If the Moore/Nemytskiǐ plane is normal then any \( f : L \to [0,1] \) extends to a continuous function from \( X \) to \([0,1] \). \( \Rightarrow \) \( |C^0(X, [0,1])| \geq |[0,1]^R| \)

On the other hand \( R = \{(x,y) \in X \mid x,y \text{ rational}\} \) is dense in \( X \) (\( \overline{\mathbb{Q}} = \mathbb{R} \)) and is countable. If \( f, g : X \to [0,1] \) are continuous and \( f|_R = g|_R \text{ then } f = g \text{ (why?)}. \Rightarrow \) \(|C^0(X, [0,1])| = |C^0(R, [0,1])| \leq |[0,1]^\mathbb{N}| \).

Contradiction since \(|[0,1]^\mathbb{N}| \) is strictly bigger than \(|[0,1]^\mathbb{N}| \).

Local compactness

Definition: A topological space is locally compact if every point has a compact neighborhood.

\( \mathbb{R} \) is locally compact but not compact: \( \forall x \in \mathbb{R} \quad [x-1, x+1] \) is a compact nbd of \( x \). Similarly \( \mathbb{R}^n \) is locally compact.
Definition A topological manifold is a topological space $X$ which is locally homeomorphic to some $\mathbb{R}^n$: for $x \in X$, there exists a open nbhd $U$ of $x$, $n \geq 0$, $V \subset \mathbb{R}^n$ open and a homeomorphism $\phi: U \to V$ ($U$, $n$, $V$ and $\phi$ depend on $x$).

Example Any manifold is locally compact. \(\text{Why?}\)

Locally compact Hausdorff (L.C.H.) spaces are particularly nice.

Lemma 19.1 Let $X$ be a locally compact Hausdorff space, $x \in X$. For any nbhd $U$ of $x$ there is a compact nbhd $N$ of $x$ with $N \subseteq U$.

Proof Since $X$ is LCH, there exists a compact nbhd $C$ of $x$. Then there exists an open nbhd $V$ of $x$ with $V \subseteq C$. Let $W = V \cap U$. Since $X$ is Hausdorff and $C$ is compact, $C$ is closed. \(\Rightarrow \overline{W} \subseteq C.\) But $\overline{W} \subseteq \overline{V}$. So $\overline{W} \subseteq C$. Since $C$ is compact and $\overline{W}$ is closed, $\overline{W}$ is compact. Since $\overline{W}$ is compact, it's regular. Regular spaces have nbhd bases of closed sets (see lecture 15). \(\Rightarrow\) There exists a nbhd $N$ of $x$, which is closed in $\overline{W}$, with $N \subseteq W$. Since $N$ is closed in $\overline{W}$ and $\overline{W}$ is compact, $N$ is compact and $x \in N \subseteq W \subseteq U$.

Remains to show: $N$ is a nbhd of $x$ in $X$.

Since $N$ is a nbhd of $x$ in $\overline{W}$, $\exists T \subseteq \overline{W}$ which is open in $\overline{W}$, with $x \in T \subseteq N$. Since $T$ is open in $\overline{W}$, $\exists O \subseteq X$ which is open in $X$, with $T = \overline{O} \cap O$.

Since $x \in N \subseteq W$, $x \in O \cap W = O \cap \overline{W} = T \subseteq N$.

Since $O \cap W$ is open in $X$ and $x \in O \cap W$, $N$ is a compact nbhd of $x$ in $X$. \(\square\)

Corollary 19.2 Suppose $X$ is LCH (locally compact Hausdorff). Then $x \in X$ and for any nbhd $U$ of $x$ there exists a open nbhd $V$ of $x$ with $V \subseteq U$ and $\overline{V}$ compact.

Proof exercise.

Theorem 19.3 An LCH space is completely regular.

Proof exercise.
Corollary 19.4. A 2\textsuperscript{nd} countable LCH space is normal and metrizable.

\textbf{Proof.} Suppose \( X \) is 2\textsuperscript{nd} countable and LCH. Then \( X \) is completely regular by 19.3, hence regular. By 18.2, \( X \) is normal. By Urysohn's metrization Thm, \( X \) is metrizable.

Compactsations.

\textbf{Definition.} A compactification of a space \( X \) is an embedding \( f: X \to Y \) so that

1) \( Y \) is compact
2) \( f(X) \) is dense in \( Y \): \( f(X) = Y \).

\textbf{Example.} \((0,1) \xrightarrow{f} [0,1] \) is a compactification.

\( g: (0,1) \to S^1 \), \( g(x) = e^{2\pi i x} \) is another compactification.

Is \( h: \mathbb{C} \to S^1 \), \( h(x) = e^{2\pi i x} \) a compactification?

\textbf{Definition.} A map \( f: X \to Y \) is a 1-point compactification if

1) \( f \) is a compactification
2) \( Y \setminus f(X) \) is a single point.