Last time: \([0,1]^\mathbb{N}\) is metrizable.

* Urysohn's metrization theorem: 2nd countable, \(T_4\), completely regular space embeds in \([0,1]^\mathbb{N}\), hence is metrizable.

Today we'd like to strengthen Urysohn's metrization theorem to:

**Theorem 18.1** 2nd countable + regular \(\Rightarrow\) metrizable.

Recall that by Urysohn's lemma, normal \(\Rightarrow\) completely regular (and \(T_4\)). Therefore, in order to prove 18.1, it is enough to prove

**Lemma 18.2** 2nd countable + regular \(\Rightarrow\) normal.

To prove 18.2 we need a definition and a lemma.

**Definition** A topological space is **Lindelöf** if every open cover has a countable subcover.

**Lindelöf's lemma** 2nd countable \(\Rightarrow\) Lindelöf.

**Proof** Let \(X\) be a 2nd countable space, \(\{B_n\}_{n \in \mathbb{N}}\) a countable basis for the topology on \(X\) and let \(\{U_x\}_{x \in A}\) be an open cover. Then

1. \(\forall x \in X \exists \alpha(x) \in N\) and \(\alpha(x) \in A\) so that \(x \in B_{n(x)} \subseteq U_{\alpha(x)}\)
2. Let \(B = \{B_n\} \subseteq \{A\} \forall x \in X \exists B_n \subseteq U_x\).

By 1, \(B\) is a countable cover of \(X\). For each \(B \in B\) choose \(\alpha(B) \in A\) so that \(B \subseteq U_{\alpha(B)}\). Then \(\{U_{\alpha(B)}\}_{B \in B}\) is a countable subcover of \(\{U_x\}_{x \in A}\).

**Proof of 18.2** Suppose \(X\) is regular, 2nd countable, \(A, B \subseteq X\) are closed and \(A \cap B = \emptyset\). Since \(X\) is regular, \(\forall x \in X\) with \(x \notin B\) \(\exists U_x, U_x'\) open so that

- \(U_x \cap B = \emptyset\)
- \(U_x' \cap A = \emptyset\)
\[
\text{set } \mathbb{U}, \mathbb{V}, \mathbb{W}, \mathbb{X}, \mathbb{Y}, \mathbb{Z}
\]

Now \( \{U_x\}_{x \in A} \cup \{X \setminus A\} \) is an open cover of \( X \).

Since \( X \) is 2nd countable, \( X \) is Lindelöf. \( \Rightarrow \) This cover has a countable subcover \( \{U_{x_n}\}_{n=1}^{\infty} \cup \{X \setminus A\} \). Then \( \{U_{x_n}\}_{n=1}^{\infty} \) is a countable cover of \( A \) with \( U_n \cap B = \emptyset \) if \( n \).

Similarly, \( \exists \) open cover \( \{V_{x_n}\}_{n=1}^{\infty} \) of \( B \) with \( V_n \cap A = \emptyset \).

Note that if \( W \subseteq X \) is open, \( C \subseteq X \) is closed. Then \( W \setminus C = W \cap (X \setminus C) \) is open.

Now let \( G_1 = U_1 \setminus V_1, G_2 = U_2 \setminus (V_1 \cup V_2), \ldots, G_n = U_n \setminus \bigcup_{k=1}^{n-1} V_k, \ldots \)

Similarly let \( H_1 = V_1 \setminus U_1, H_2 = V_2 \setminus (U_1 \cup U_2), \ldots, H_n = V_n \setminus \bigcup_{k=1}^{n-1} U_k, \ldots \)

Then \( G_i, H_i \) are open for all \( i \in \mathbb{N} \). Let \( G = \bigcup_{i=1}^{\infty} G_i, H = \bigcup_{j=1}^{\infty} H_j \); they are open.

Since \( V_n \cap B = \emptyset \) for \( n \) and since \( \bigcup_{n=1}^{\infty} U_n \supseteq A \), \( G \supseteq A \). Similarly, \( B \subseteq H \).

We now argue that \( G \cap H = \emptyset \).

Suppose not: \( G \cap H \neq \emptyset \). Then \( \exists z \in G \cap H \Rightarrow \exists n, m \in \mathbb{N} \) s.t. \( z \in G_n \cap H_m \).

We may assume \( n \neq m \). Since \( U_m \subseteq V_m \) and \( G_n = U_n \setminus \bigcup_{k=1}^{n-1} V_k \), \( H_m \cap G_n \subseteq V_m \cap G_n = \emptyset \).

Consider \( H_m \cap V_m \) and \( G_n \subseteq U_n \setminus \bigcup_{k=1}^{n-1} V_k \). Then \( H_m \cap G_n = \emptyset \).

Contradiction: \( \forall z \in G \cap H \).

We conclude that the space \( X \) is normal. \( \square \)

This proves 18.2 and therefore 18.1: 2nd countable + regular \( \Rightarrow \) metrizable.

Our next goal:

**Tietze extension theorem** Let \( X \) be a normal topological space, \( F \subseteq X \) a closed subset, \( f: F \rightarrow [0,1] \) a continuous function. Then \( \exists \) a continuous function \( \tilde{f}: X \rightarrow [0,1] \) an extension of \( f \), so that \( \tilde{f}|_F = f \).

**Remarks**

1) The requirement that \( F \) is closed in Tietze extension theorem is essential:

Consider \( X = \mathbb{R} \) with the standard topology, \( A = \mathbb{R} \setminus \{0\} \). Consider \( f: A \rightarrow [0,1] \)
defined by \( f(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases} \). Then \( f \) is continuous on \( A \), but \( \tilde{f} : \mathbb{R} \to [0,1] \) is not continuous on \( A \), but \( \tilde{f} \) is continuous and \( \tilde{f} |_{\mathbb{R} \setminus \{0\}} = f \).

2) One can use Tietze extension theorem to prove that Moore-Nemytski plane is not normal. We'll come back to that.

**Definition** Let \( X \) be a topological space, \((Y,d)\) a metric space. A sequence of functions \( \{f_n : X \to Y\}_{n=1}^\infty \) converges uniformly to \( f : X \to Y \) if \( \forall \varepsilon > 0 \exists N \in \mathbb{N} \) so that \( n \geq N \Rightarrow d(f_n(x), f(x)) < \varepsilon \) for all \( x \in X \).

The following result may be familiar from analysis:

**Lemmal8.8** Suppose \( X \) be a space, \((Y,d)\) a metric space and \( \{f_n : X \to Y\}_{n=1}^\infty \) a sequence of continuous functions that converge uniformly to \( f : X \to Y \). Then \( f \) is continuous.

**Proof** To prove that \( f \) is continuous, given \( x_0 \in X \) and \( \varepsilon > 0 \) we need to find a nbd \( U \) of \( x_0 \) so that \( x \in U \Rightarrow d(f(x), f(x_0)) < \varepsilon \).

Since \( f_n \to f \) uniformly \( \exists N \in \mathbb{N} \) so that \( n \geq N \Rightarrow d(f_n(x), f(x)) < \varepsilon/3 \) for all \( x \in X \).

Since \( f_n \) is continuous at \( x_0 \), \( \exists \) a nbd \( U \) of \( x_0 \) so that \( x \in U \Rightarrow d(f_n(x), f_n(x_0)) < \varepsilon/3 \).

Therefore, \( \forall x \in U \)
\[
d(f(x), f(x_0)) \leq d(f(x), f_n(x)) + d(f_n(x), f_n(x_0)) + d(f_n(x_0), f(x_0))
\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\]

**Proof of Tietze extension theorem** We may assume \( 0 = \inf_{x \in F} f(x), \ 1 = \sup_{x \in F} f(x) \). Let \( A = f^{-1}([0,1/3]), \ B = f^{-1}(1/3,1] \). Then \( A,B \) are closed in \( F \), disjoint and nonempty. Since \( F \) is closed, \( A,B \) are closed in \( X \).

By Urysohn's lemma \( \exists g : X \to [0,1/3] \) continuous with \( g|_A = 0, \ g|_B = 1/3 \). Then \( x \in F \Rightarrow f(x) \leq 1/3 \Rightarrow g(x) = 0 \).
and \( x \in F \), \( f(x) \geq 2/3 \) \( \Rightarrow \) \( g_1(x) = \frac{1}{3} \).

Let \( f_1 = f - g_1 \). Then \( f_1: F \rightarrow \mathbb{R} \) is continuous and \( 0 \leq f_1(x) \leq \frac{2}{3} \).

Now repeat the construction with \( f \) replaced by \( f_1 \): we get \( g_2: X \rightarrow [0, \frac{1}{3}, \frac{2}{3}] \)

continuous so that for \( x \in F \)

\[
 f_1(x) \leq \frac{1}{3} \cdot \frac{2}{3} \Rightarrow g_2(x) = 0 \quad \text{and} \quad f_1(x) \geq \frac{2}{3} \cdot \frac{2}{3} \Rightarrow g_2(x) = \frac{1}{3} \cdot \frac{2}{3} .
\]

Let \( f_2 = f_1 - g_2 \). Then \( 0 \leq f_2(x) \leq \frac{2}{3} \).

Inductive step: Suppose we have defined \( f_n : F \rightarrow \mathbb{R} \), continuous, with \( 0 \leq f_n(x) \leq \left( \frac{2}{3} \right)^n \).

Then \( \exists g_n : X \rightarrow [0, \frac{1}{3}, \frac{2}{3}] \), continuous so that \( \forall x \in F \)

\[
 f_n(x) \leq \frac{1}{3} \left( \frac{2}{3} \right)^n \Rightarrow g_{n+1}(x) = 0 \quad \text{and} \quad f_n(x) \geq \frac{2}{3} \left( \frac{2}{3} \right)^n \Rightarrow g_{n+1}(x) = \frac{1}{3} \left( \frac{2}{3} \right)^n .
\]

Let \( f_{n+1} = f_n - g_{n+1} \). Then \( 0 \leq f_{n+1}(x) \leq \left( \frac{2}{3} \right)^{n+1} \).

Since \( 0 \leq g_n(x) \leq \frac{1}{3} \left( \frac{2}{3} \right)^{n-1} \), the series \( g(x) = \sum_{n=1}^{\infty} g_n(x) \) converges uniformly on \( X \).

By 18.3, \( g \) is continuous on \( X \). Also \( \forall x \in F \)

\[
 f(x) - g_1(x) = f_1(x)
\]

\[
 f_1(x) - g_2(x) = f_2(x)
\]

\[
 \vdots
\]

\[
 f_{n-1}(x) - g_n(x) = f_n(x)
\]

\[
 \Rightarrow (\forall) \quad f(x) - (g_1(x) + \ldots + g_n(x)) = f_n(x) \quad \forall x \in F
\]

Since \( 0 \leq f_n(x) \leq \left( \frac{2}{3} \right)^n \), (\( \star \)) \( \Rightarrow \) \( 0 = \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \left( f(x) - \sum_{k=1}^{n} g_n(x) \right) = f(x) - g(x) \).

\[
 \therefore \quad f(x) = g(x) \quad \forall x \in F
\]

i.e. \( g: X \rightarrow [0,1] \) is a desired extension of \( f \). \( \square \)