Last time: • a metric space is compact ⇔ every sequence has a convergent subsequence ⇔ the space is complete and totally bounded.
• defined $T_0, T_1, T_2$ (Hausdorff), $T_3$ (regular) and $T_4$ (normal)

**Proposition 15.1** A Hausdorff space $X$ is regular ⇔ for every point $x \in X$ a nbd $N$ of $x$ contains a closed nbd of $x$.

**Proof $(\Rightarrow)$** Suppose $X$ is a regular space, $x \in X$ and $N$ is a nbd of $x$. Then:

1. Open set $V$ with $x \in V \subseteq N$. $C = X \setminus V$ is closed and $x \notin C$. Since $X$ is regular, there exists an open nbhd $W$ of $x$ with $U \cap W = \emptyset$. Then $U \subseteq X \setminus W \subseteq X \setminus C = V \subseteq N$.

$\Rightarrow$ $X \setminus W$ is a desired closed nbd of $x$.

**Proof $(\Leftarrow)$** Suppose $x \in X$, $C \subseteq X$ is closed and $x \notin C$. Then $X \setminus C$ is an (open) nbd of $x$.

By assumption, there exists a closed nbd $N$ of $x$ with $N \subseteq X \setminus C$. Since $N$ is a nbd of $x$:

1. Open nbd $U$ of $x$ with $U \subseteq N$. Since $N$ is closed, $V = X \setminus N$ is open. Also $V = X \setminus N = X \setminus (X \setminus C) = C$. Finally, since $U \subseteq N$ and $V = X \setminus W$, $U \cap V = \emptyset$.

Therefore $X$ is regular.

**Exercise** A subspace of a regular space is regular.

There are Hausdorff spaces that are not $T_3$. We now construct an example of one.

**Note:** If $(X, T)$ is Hausdorff and $T'$ is another topology with $T \subseteq T'$ Then $(X, T')$ is Hausdorff.

Now let $(X, T) = (\mathbb{R}, T_{\text{standard}})$. Let $T'$ be the smallest topology on $\mathbb{R}$ containing $T_{\text{standard}}$ and the set $\mathbb{R} \setminus K$ where $K = \{ \frac{1}{n} \mid n \in \mathbb{N} \}$:

$$T' = \langle T_{\text{standard}} \cup \{ \mathbb{R} \setminus K \} \rangle.$$  

**Note that** $B = \{ (a, b) \mid a, b \in \mathbb{R}, a < b \} \cup \{ (c, d) \setminus K \mid c, d \in \mathbb{R}, c < d \}$ is a basis of $T'$.

$T'$ is a Hausdorff topology: we can use standard open sets to separate points. However $T'$ is not regular: $K$ is closed in the $T'$ topology, $0 \notin K$ but there is no way to separate $0$ and $K$, which we will prove.
Note that $y \in \mathbb{R}, y > 0, \exists n_x \in \mathbb{N}$ s.t. $0 < \frac{1}{n_x} < x$.

If $U$ is a nbd of $0$ in $T'$, then either $\exists a, b$ with $a < 0 < b$ and $(a, b) \subseteq U$ or $\exists c, d \subseteq \mathbb{R}$ with $c < 0 < d$ and $(c, d) \setminus K \subseteq U$.

Suppose $(a, b) \subseteq U$. Since $0 < b$, $\exists n_b \in \mathbb{N}$ s.t. $\frac{1}{n_b} \notin (a, b) \Rightarrow U \cap (a, b) \cap K \neq \emptyset$.

Suppose $(c, d) \setminus K \subseteq U$. Since $0 < d$, $\exists n_d \in \mathbb{N}$ s.t. $0 < \frac{1}{n_d} < d$. Then any open nbd $V$ of $K$ contains $\frac{1}{n_d} \Rightarrow V \cap (c, d) \setminus K \neq \emptyset$.

$\therefore$ $0$ and $K$ cannot be separated by sets in $T'$.

There are also regular spaces that are not normal, so $T_3 \not\subseteq T_4$.

Constructing them is not hard. Proving that they are regular and not normal is work.

Example 15.2 Consider $\mathbb{R}$ with the topology generated by half-closed intervals $[a, b)$, $a < b$.

Denote this space by $\mathbb{R}_e$. One can show:

- $\mathbb{R}_e$ is normal, hence regular
- A product of two regular spaces is regular, hence $\mathbb{R}_e^2 = \mathbb{R}_e \times \mathbb{R}_e$ is regular.
- $\mathbb{R}_e^2$ (which is called Sorgenfrey plane) is not normal.

Example 15.3 Moore/Niemytskii plane.

Let $\Gamma = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$. For all $(x, y) \in \Gamma$ let $U_{x,y} = \{(x, y) \cup B\}$ By $(x, y)$:

\[ U_{x,y} \]

Let $\mathcal{T} = \{T_{\text{standard}} \cup \{U_{x,y} \mid (x, y) \in \Gamma\} \}$

Then $(\Gamma, \mathcal{T})$ is regular and not normal.

Theorem 15.4 Every metric space is normal.

Proof Let $(X, d)$ be a metric space, $A, B \subseteq X$ closed subsets s.t. $A \cap B = \emptyset$.

Since $B$ is closed and $A \cap B = \emptyset$, $\forall a \in A \exists \varepsilon_a > 0$ s.t. $B_{\varepsilon_a}(a) \cap B = \emptyset$. 

Similarly \( \forall b \in B \exists \varepsilon > 0 \text{ s.t. } B_{\varepsilon b}(b) \cap A = \emptyset \).

Let \( U = \bigcup_{a \in A} B_{\varepsilon a/2}(a) \), \( V = \bigcup_{b \in B} B_{\varepsilon b/2}(b) \).

If \( z \in U \cap V \), then \( \exists a \in A, b \in B \text{ s.t. } z \in B_{\varepsilon a/2}(a) \cap B_{\varepsilon b/2}(b) \).

We may assume: \( \varepsilon a \leq \varepsilon b \). Then
\[
d(a, b) = d(a_1, z) + d(z, b) < \varepsilon a/2 + \varepsilon b/2 < \varepsilon b \Rightarrow a \in B_{\varepsilon b}(b). \]

But \( B_{\varepsilon b}(b) \cap A = \emptyset \). Contradiction. So \( U \cap V = \emptyset \).

Therefore, \( U, V \) are two open sets that separate \( A \) and \( B \).

**Corollary 15.5:** If \((X, T)\) is not a normal topological space then \((X, T)\) is not metrizable: \( \exists \) a metric \( d \) with \( T_d = T \).

**Next time:** Compact Hausdorff spaces are normal.