Last time: defined nets and their convergence
- proved that for $A \subseteq X$ (X top space), $y \in A \iff \exists$ a net in A that converges to y.
- $f : X \to Y$ is continuous $\iff$ for net $(x_\lambda)_{\lambda \in \Lambda}$ in X with $x_\lambda \to w$, $f(x_\lambda) \to f(w)$.
- $X$ in Hausdorff $\iff$ limits of nets in X are unique.

Subnets to nets are like subsequences to sequences.

**Definition 10.3** Let $\Lambda, M$ be two directed sets and $x : \Lambda \to X$, $x \mapsto x_\lambda$, a net in $X$.

(i) A function $\psi : M \to \Lambda$ is nondecreasing if $\mu_1, \mu_2 \in M$ and $\mu_1 < \mu_2$ implies $\psi(\mu_1) < \psi(\mu_2)$.

(ii) A function $\psi : M \to \Lambda$ is cofinal if $\forall \lambda \in \Lambda \exists \mu \in M$ such that $\lambda < \psi(\mu)$.

A subnet of $x : \Lambda \to X$, $x \mapsto x_\lambda$, in the composite $x \circ \psi : M \to X$, $m \mapsto x_{\psi(m)}$, so that $\psi$ is nondecreasing and cofinal.

**Proposition 10.2** Let $(x_\lambda)_{\lambda \in \Lambda}$ be a net in a topological space $X$ that converges to $w \in X$.

Then any subnet $(x_{\psi(m)})_{m \in M}$ of $(x_\lambda)$ also converges to $w$.

**Proof** Let $N$ be a neighborhood of $w$. Since $x_\lambda \to w$, $\exists \lambda_0 \in \Lambda$ so that $x_\lambda \in N$ for all $\lambda$ with $\lambda_0 < \lambda$.

Since $\psi$ is cofinal, $\exists \mu \in M$ such that $\lambda_0 < \psi(\mu)$.

Then for any $\mu \in M$ with $\mu_0 < \mu$, $\psi(\mu_0) < \psi(\mu)$ (since $\psi$ is nondecreasing).

Therefore $\mu_0 < \mu \Rightarrow \lambda_0 < \psi(\mu_0) < \psi(\mu) = x_\psi(\mu) \in N$, so $x_\psi(\mu) \to w$.

**Definition 10.3** Let $X$ be a topological space. A collection $\{U_\alpha\}_{\alpha \in A}$ of subsets of $X$ is a cover of $X$ if $\bigcup_{\alpha \in A} U_\alpha = X$.

$\{U_\alpha\}_{\alpha \in A}$ is an open cover if $\{U_\alpha\}_{\alpha \in A}$ is a cover and each $U_\alpha$ is open.

A subcover of $\{U_\alpha\}_{\alpha \in A}$ is a subcollection $\{U_\beta\}_{\beta \in B}$ for some $B \subseteq A$ so that $\bigcup_{\beta \in B} U_\beta = X$.

**Example** Let $(X, d)$ be a metric space. The set $\{B_r(x) : r < \epsilon \}$ of open balls is
a cover of $X$, $(B_n)_{n \in \mathbb{N}} \times (B_m)_{m \in \mathbb{N}}$ is a subcover of $(B_n)_{n \in \mathbb{N}} \times (B_m)_{m \in \mathbb{N}}$.}

**Definition 10.4** A topological space $X$ is compact if every open cover $(U_a)_{a \in A}$ has a finite subcover $(U_{a_1}, \ldots, U_{a_k})$ (some $k \geq 0$, some $a_1, \ldots, a_k \in A$).

We'll prove:

1. $X$ is compact $\iff$ every net in $X$ has a convergent subnet.
2. $K \subseteq \mathbb{R}^n$ is compact $\iff$ $K$ is closed and bounded (i.e., $\exists r > 0$ s.t. $K \subseteq B_r(0)$).
3. Tychonoff's Theorem: A product of compact spaces is compact.

and a few more results.

**Proposition 10.5 (technical)** Let $X$ be a topological space, $Y \subseteq X$ a subspace (i.e., a subset given the subspace topology). $Y$ is compact $\iff$ it collection $(U_a)_{a \in A}$ of open sets open in $X$ so that $Y \subseteq \bigcup_{a \in A} U_a$ and $a_1, \ldots, a_k \in A$ s.t. $Y \subseteq U_{a_1} \cup \ldots \cup U_{a_k}$.

**Proof (\(\Rightarrow\))** $(U_a)_{a \in A}$ is an open cover of $Y$. Since $Y$ is compact, $\exists a_1, \ldots, a_k \in A$ s.t.

$$Y = (U_{a_1} \cup \ldots \cup U_{a_k}).$$

\(\Leftarrow\) Suppose $(U_a)_{a \in A}$ is an open cover of $Y$. By definition of subspace topology, $(U_a)_{a \in A}$ exists in $X$ open s.t. $V_a = Y \cap U_a$. And then $Y = \bigcup_{a \in A} V_a$. By assumption $\exists a_1, \ldots, a_k \in A$ s.t. $Y \subseteq U_{a_1} \cup \ldots \cup U_{a_k}.

\Rightarrow Y \subseteq Y \cap (U_{a_1} \cup \ldots \cup U_{a_k}) = (Y \cap U_{a_1}) \cup \ldots \cup (Y \cap U_{a_k}) = V_{a_1} \cup \ldots \cup V_{a_k}.

Hence $Y = V_{a_1} \cup \ldots \cup V_{a_k}$ and therefore $Y$ is compact.

**Lemma 10.6 (important!)** Images of compact spaces under continuous maps are compact.

**Proof** Suppose $f: X \to Y$ is continuous, $X$ is compact. We'd like to prove: $f(X)$ is compact (where $f(X)$ is given a subspace topology).

By 10.5, enough to show: $A$ collection $(U_a)_{a \in A}$ of open sets in $Y$ with
Since $f$ is continuous and since $f(X) \subseteq \bigcup_{x \in A} f^{-1}(U_x)$ is an open cover of $X$, Since $X$ is compact, there exist $x_1, \ldots, x_n$ such that $X \subseteq \bigcup_{i=1}^n f^{-1}(U_{x_i})$. Thus $f(X) \subseteq U_{x_1} \cup \cdots \cup U_{x_n}$.

Lemma 10.7 A closed subset of a compact topological space is compact.

Proof Let $K$ be a closed subset of a compact topological space $X$, let $\{U_x\}_{x \in A}$ be a collection of open subsets of $X$ with $K \subseteq \bigcup_{x \in A} U_x$. Then $\{U_x\}_{x \in A} \cup \{X \setminus K\}$ is an open cover of $X$.

Since $X$ is compact, there exist $x_1, \ldots, x_n$ such that $X = U_{x_1} \cup \cdots \cup U_{x_n} \cup (X \setminus K)$. Thus $K \subseteq U_{x_1} \cup \cdots \cup U_{x_n}$.

Lemma 10.8 A compact subset of a Hausdorff space is closed.

Proof Let $X$ be a Hausdorff space. Suppose $K \subseteq X$ is compact. We'd like to prove that $X \setminus K$ is open: $\forall x \in X \setminus K \exists$ open nbhd $U_x$ of $x$ s.t $U \cap K = \emptyset$.

Fix $x \in X \setminus K$. Since $X$ is Hausdorff, for $k \in K$ there exist open nbhd $U_k$ of $x$, $V_k$ of $k$ so that $U_k \cap V_k = \emptyset$. By construction, $K \subseteq \bigcup_{k \in K} U_k$. Since $K$ is compact, there exist $k_1, \ldots, k_n \in K$ so that $K \subseteq V_{k_1} \cup \cdots \cup V_{k_n}$ (we're using 10.5 again).

Now let $U = U_{k_1} \cup \cdots \cup U_{k_n}$. Then $U \cap V_{k_i} = \emptyset$ for $i = 1, \ldots, n$.

Thus $U \cap K = \emptyset$, and we're done.

WARNING If $X$ is not Hausdorff, a compact subset $K \subseteq X$ need not be closed.

Ex $X = \{a, b, c\}$, $\mathcal{T} = \{\emptyset, X, \{a, b\}, \{b, c\}, \{b, c\}\}$. $K = \{b, c\}$ is compact (Why?) but not closed (why?).

Ex Let $Y = \mathbb{R} \times \{0\} \cup \mathbb{R} \times \{1\} \subseteq \mathbb{R}^2$. Let $\sim$ be the smallest equivalence relation on $Y$ so that $(x, 0) \sim (x, 1)$ for all $x \neq 0$. Let $Q = Y / \sim$; this is the "line with
Give $Q$ the quotient topology so that the quotient map $g: Y \rightarrow Q$ is continuous. We'll prove: $[0,1] \subseteq R$ is compact. Assume this fact for now.

Now $R \subseteq Y$, $y(x) = (x, 0)$ is continuous. Hence $g \circ y: [0,1] \rightarrow Q$ is continuous. By 10.6, $K = (g \circ y)([0,1]) \subseteq Q$ is compact, but it's not closed: every open neighborhood of $[(x,1)]$ intersects $K$ nontrivially, so $[(0,1)] \notin K$.

The following result is very useful

**Lemma 10.9** Let $X$ be compact, $Y$ Hausdorff and $f: X \rightarrow Y$ a continuous bijection. Then $f$ is a homeomorphism.

Proof. We want to show that $g = f^{-1}: Y \rightarrow X$ is continuous. Enough to show:

- If $C \subseteq X$ closed, $g^{-1}(C)$ is closed (why?). Note: $g^{-1}(C) = f(C)$.
- Since $X$ is compact and $C \subseteq X$ is closed, $C$ is compact.
- Since $f$ is continuous and $C$ is compact, $f(C)$ is compact.
- Since $Y$ is Hausdorff and $f(C)$ is compact, $f(C)$ is closed. \qed