Last time we defined a natural topology on binary products: $\mathcal{Y}$.

Given $(X, T_X), (Y, T_Y)$, the product topology on $X \times Y$ is the smallest topology $T_{X \times Y}$ making the projections $p_X : X \times Y \to X$, $p_Y : X \times Y \to Y$ continuous.

The product topology generalizes to arbitrary products.

Recall if $\prod_{x \in A} X_x$ is a family of sets, the product $\prod_{x \in A} X_x$ is a set that is usually defined by a construction:

$$\prod_{x \in A} X_x = \{ f : A \to \bigcup_{x \in A} X_x \mid f(x) \in X_x \}.$$ 

$$\begin{align*}
\exists x & \quad A = \{1, 2\} \\
\prod_{x \in \{1, 2\}} X_x & = \{ f : \{1, 2\} \to X_1 \times X_2 \mid f(1) \in X_1, f(2) \in X_2 \} \\
\text{bijective} & \quad X_1 \times X_2 \\
\downarrow & \quad (f(1), f(2)) \\
(a_1 \leftarrow 1) & \quad \leftrightarrow (a_1, a_2) \\
(a_2 \leftarrow 2) & \quad \leftrightarrow (a_1, a_2)
\end{align*}$$

It comes with canonical projections $p_\beta : \prod_{x \in A} X_x \to X_\beta$

$$p_\beta(f) = f(\beta)$$

And it has a universal property: given any family $A$ maps $(g_\alpha : X \to X_\alpha)$

$$\exists ! g : X \to \prod_{x \in A} X_x$$

$$p_\alpha \circ g = g_\alpha.$$ 

Namely, take

$$g(z) : A \to \bigcup_{x \in A} X_x$$

is given by

$$(g(z))(x) = g_\alpha(z).$$
Conversely, we could have defined \( \prod X_a \) as a set with a family of maps \( \pi_p : \prod X_a \to X_p \) having the universal property. One can show: given any two such sets, there is a unique bijection between them.

**Now topology:** let \( \{(X_a, \mathcal{T}_a)\}_{a \in A} \) be a family of topological spaces. The **product topology** \( \mathcal{T} \) on \( \prod X_a \) is the smallest topology so that the projections \( \pi_p : (\prod X_a, \mathcal{T}) \to (X_p, \mathcal{T}_p) \) are continuous.

Easy to see:

\[
\mathcal{B} = \{ \cap_i \pi_i^{-1} (U_{a_i}) \mid U_{a_i} \in \mathcal{T}_{a_i} \}
\]

is a basis for \( \mathcal{T} \).

**Universal property?**

Suppose we have a family of continuous maps

\[
\{ g_a : (Z, \mathcal{T}_Z) \to (X_a, \mathcal{T}_{a}) \}_{a \in A}
\]

Then \( \prod \) set theoretic map \( g : Z \to \prod X_a \) with \( \pi_a \circ g = g_a \).

Is \( g \) continuous? Yes, since \( \forall a \forall U_a \in \mathcal{T}_{a} \)

\[
g^{-1}( \pi_a^{-1} (U_a)) = (\pi_a \circ g)^{-1} (U_a) = g_a^{-1} (U_a) \in \mathcal{T}_Z
\]

and

\[
\mathcal{B} = \{ \pi_i^{-1} (U_a) \mid a \in A, U_a \in \mathcal{T}_{a} \} \text{ is a subbase for the product topology.}
\]
Remark: The box topology \( \tau_{\text{box}} \) on \( \prod_{\alpha \in A} X_{\alpha} \) is the topology with the basis:

\[
\mathcal{B}_{\text{box}} = \{ \prod_{\alpha \in A} U_{\alpha} \mid U_{\alpha} \subseteq X_{\alpha} \text{ is open in } X_{\alpha}, \ \alpha \in A \}.
\]

\( \tau_{\text{box}} \neq \tau_{\text{product}} \) if the product is infinite.

Note: If \( (X_{\alpha}, \tau_{\alpha}) = (X_{\alpha}, \tau_{\alpha}) \) for all \( \alpha \),

we have the diagonal map

\[
\Delta : X \to \prod_{\alpha \in A} X_{\alpha}
\]

uniquely defined by

\[
(\pi_{\alpha} \circ \Delta)(x) = x, \quad \forall \alpha,
\]

i.e., \( \Delta : x \mapsto (x_1 \mapsto x, \forall \alpha) \)

or "\( \Delta(x) = (x, x, \ldots) \)".

\[
\Delta^{-1}(\prod_{\alpha \in A} U_{\alpha}) = \bigcap_{\alpha \in A} U_{\alpha},
\]

which need not be open.

\( \Delta \) need not be continuous with respect to box topology.

Aside: The notion dual to that of product is disjoint union: given a family of sets \( \{X_{\alpha}\}_{\alpha \in A} \),

\[
\prod_{\alpha \in A} X_{\alpha} := \bigcup_{\alpha \in A} X_{\alpha} \times \{\alpha\}.
\]

\( \prod_{\alpha \in A} X_{\alpha} \) comes with canonical inclusions

\[
\iota_{\alpha} : X_{\alpha} \to \prod_{\alpha \in A} X_{\alpha},
\]

\[
\iota_{\alpha}(x) = (x, \alpha)
\]

Universal property:

Given any family \( \{h_{\alpha} : X_{\alpha} \to Z\} \),

\[
\exists ! \quad h : \prod_{\alpha \in A} X_{\alpha} \to Z \quad \text{with}
\]

\[
\iota_{\alpha} \circ h = h_{\alpha}
\]

(prove it)
Definition 4.1. A map \( f : (X, T_X) \to (Y, T_Y) \) is open if 
\[ \forall U \in T_X, \ f(U) \in T_Y. \] 
I.e., \( f \) maps open sets to open sets.

Prop 4.2. Let \( \{ (X_{a}, T_{a}) \}_{a \in A} \) be a family of top spaces. 
Endow \( \prod_{a \in A} X_{a} \) with product topology. Then the projection \( p_{b} : (\prod_{a \in A} X_{a}, T_{\text{product}}) \to (X_{b}, T_{b}) \) are open maps.

Proof. Product is generated by the sets of the form \( p_{a}^{-1}(U_{a}) \), \( U_{a} \in T_{a} \).

\[
p_{b}(p_{a}^{-1}(U_{a})) = \begin{cases} X_{b} & \text{if } a = b \\ U_{a} & \text{if } a \neq b \end{cases}
\]

Similarly,

\[
p_{b}(\bigcap_{a \in A} p_{a}^{-1}(U_{a})) = \bigcap_{a \in A} U_{a} \quad \text{for some } i.
\]

\[ \Rightarrow p_{b}(\bigcup_{a \in A} p_{a}^{-1}(U_{a})) = \bigcup_{a \in A} p_{b}(p_{a}^{-1}(U_{a})) = \bigcup_{a \in A} p_{a}^{-1}(U_{a}) \]

Quotient topology

Ex. A circle \( S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \) can also be thought of as \([0, 2\pi] \) with end points identified.

\[ \mathbb{R}/\mathbb{Z} = [0, 2\pi] \quad \text{where } x \sim x' \iff x - x' \in \mathbb{Z} \]

\[ \mathbb{R}/2\pi\mathbb{Z} = \mathbb{R}/\mathbb{Z} \quad \text{when } x \sim x' \iff x - x' \in 2\pi\mathbb{Z} \]

Since \( S^1 \subset \mathbb{R}^2 \), it has a subspace topology.

We want \( [0, 2\pi] / \sim \) to get its topology from \([0, 2\pi]\)

and \( \mathbb{R}/\mathbb{Z} \) to get its topology from \( \mathbb{R} \)

And it should be same topology.
Last time arbitrary products of topological spaces

\[ \prod_{x \in A} (X_x, T_x) = (\prod_{x \in A} X_x, \text{product}) \]

Today Quotient topology,

Motivating (?) example: A circle \( S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \) has a topology induced by the inclusion \( S^1 \hookrightarrow \mathbb{R}^2 \), where \( \mathbb{R}^2 \) is given the standard topology (the one induced by the Euclidean metric).

But we can also think of the circle \( S^1 \) as \([0, 2\pi] \) (topology induced by \([0, 2\pi] \hookrightarrow \mathbb{R} \)) with end points identified, ie \( S^1 \cong [0, 2\pi]/\sim \) where on \( 2\pi \)

Is there a natural way to give the quotient \([0, 2\pi]/\sim \) a topology so that the circle gets the "right" topology?

Aside \([0, 2\pi]/\sim \) and \( S^1 \subseteq \mathbb{R}^2 \) are not the same set. So what could we mean by

\[ S^1 \subseteq [0, 2\pi]/\sim \] as top spaces?

Def A continuous map \( f : (X, T_X) \to (Y, T_Y) \) is a homeomorphism (an isomorphism in the category of topological spaces) if there is a continuous map \( g : (Y, T_Y) \to (X, T_X) \) so that \( fog = id_Y \) and \( gof = id_X \).

Exercise \( f : (X, T_X) \to (Y, T_Y) \) is a homeomorphism if \( f \) is a continuous open bijection.
**Warning** Not all continuous bijections are open:

Consider \( f : [0, 2\pi) \to S^1 \)

\[ f(t) = (\cos t, \sin t) \]

It is a continuous bijection, but \( f([0, \pi)) \) is not open in \( S^1 \).

---

**General Question** \((X, T_X)\) top space, \(\sim\) an equiv relation on \(X\). What's the "right" topology on \(X/\sim\)?

Given a set \(X\) and an equivalence relation \(\sim\) on \(X\) we have a canonical projection \(\pi : X \to X/\sim\), \(\pi(x) = \) equiv class of \(x\).

**Universal property:**

\(\forall f : X \to Z\) with \(f\) constant on equiv classes of \(\sim\) (i.e. \(x \sim x' \Rightarrow f(x) = f(x')\))

\(\exists! \overline{f} : X/\sim \to Z\) so that

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Z \\
\downarrow \pi & & \downarrow \overline{f} \\
X/\sim & \xrightarrow{\overline{f}} & Z
\end{array}
\]

This suggests:

If \((X, T_X)\) top space, \(\sim\) equiv relation, the quotient topology \(T_{\text{quotient}}\) on \(X/\sim\) is the topology with the following two properties:

1) \(\pi : (X, T_X) \to (X/\sim, T_{\text{quotient}})\) is continuous

2) \(\forall\) continuous \(f : (X, T_X) \to (Z, T_Z)\) which is constant on equiv classes of \(\sim\),

\(\overline{f} : X/\sim \to Z\), \(\overline{f}(\pi(x)) = f(x)\)

is continuous.
Prop 5.1

Let \((X, T_X)\), \(\sim\) be as above.

Then the collection \(\mathcal{T}\) of subsets of \(X/\sim\) be fixed by

\[
\mathcal{T}_{\text{quot}} = \{ U \subseteq X/\sim \mid \pi^{-1}(U) \in T_X \}
\]

is a quotient topology.

Proof. We first check that \(\mathcal{T}_{\text{quot}}\) is a topology:

- \(\pi^{-1}(\emptyset) = \emptyset \Rightarrow \emptyset \in \mathcal{T}_{\text{quot}}\)
- \(\pi^{-1}(X/\sim) = X \subseteq T_X \Rightarrow X/\sim \in \mathcal{T}_{\text{quot}}\)
- \(\pi^{-1}(U \cap V) = \pi^{-1}(U) \cap \pi^{-1}(V) \in \mathcal{T}_{\text{quot}}\)

So \(U, V \in \mathcal{T}_{\text{quot}} \Rightarrow \pi^{-1}(U \cap V) \in \mathcal{T}_{\text{quot}}\).

Finally, if \(U, \alpha \in T_{\text{quot}}\),

\[
\pi^{-1}(U \cup U_{\alpha}) = U \cup \pi^{-1}(U_{\alpha}) \in T_X \\
\Rightarrow U \cup U_{\alpha} \in T_{\text{quot}}.
\]

- By definition of \(T_{\text{quot}}\), \(\pi : X \to X/\sim\) is continuous.
- Universal property? Suppose \(f : (X, T_X) \to (Z, T_Z)\) is continuous and factors (as a map of sets) as

\[
f = f_T \circ \pi,
\]

Is \(\bar{f} : (X/\sim, T_{\text{quot}}) \to (Z, T_Z)\) continuous?

\[
\forall U \in T_Z, \pi^{-1}(f^{-1}(U)) = (f \circ \pi)^{-1}(U) = f^{-1}(f(\pi^{-1}(U))) \subseteq T_X
\]

since \(f\) is continuous. By def of \(T_{\text{quot}}\), \(f^{-1}(U) \in T_{\text{quot}}\).

\[
\Rightarrow \bar{f} \text{ is continuous}.
\]

Corollary 5.2

Suppose \((X, T_X)\) a top space, \(\sim\) an equiv relation

\((Y, T_Y)\) top space, \(q : (X, T_X) \to (Y, T_Y)\)
a surjective continuous map with \( \bar{q}(q(u)) = Tx \) for all \( u \in X \).

Suppose further that \( \forall U \subseteq Y \)

\[ U \subseteq \bar{Y} \iff q'(U) \subseteq \bar{X}. \]

Then \( (Y, \bar{Y}) \) is homeomorphic to \( (X/\sim, T_{\text{quot}}) \).

**Remark**: such \( q : (X, TX) \rightarrow (Y, \bar{Y}) \) is called an identification map.

**Proof**: Since \( q \) is constant on equivalence classes of \( \sim \) and \( (X/\sim, T_{\text{quot}}) \) has the universal property,

\[ \bar{q} : (X/\sim, T_{\text{quot}}) \rightarrow (Y, \bar{Y}) \]

is a continuous bijection. Hence, since

\[ T_x \in q^{-1}(U) = ((q \circ q')^{-1}(U)) = q^{-1}(\bar{q}(U)), \]

By the assumption on \( \bar{Y} \),

\[ (\bar{q}^{-1})'(U) = \bar{q}(U) \subseteq \bar{Y}, \]

\[ \Rightarrow \bar{q}^{-1} : (Y, \bar{Y}) \rightarrow (X, T_{\text{quot}}) \text{ is continuous} \]

\[ \Rightarrow \bar{q} : (X, T_{\text{quot}}) \rightarrow (Y, \bar{Y}) \]

is a homeomorphism.

---

**Moral**: Anything that acts as a quotient \( \sim \) (homeomorphic to) a quotient.

\[ \exists f : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f(x, y) = x \]

\[ \forall U \subseteq \mathbb{R}, \quad f'(U) = U \times \mathbb{R} \]

\( U \subseteq \mathbb{R} \) open \( \iff U \times \mathbb{R} \subseteq \mathbb{R}^2 \) is open.

\[ \Rightarrow \mathbb{R}^2/\sim \text{ where } \sim (x, y) \sim (x, y') \forall x, y, y' \]

is homeomorphic to \( \mathbb{R} \).
Closure, limit points.

**Def 6.1** A neighborhood $N$ of a point $x_0$ in a top space $(X, T_X)$ is a subset $N$ of $X$ such that $\exists$ open set $U \subseteq T_X$ with $x_0 \in U \cap N$.

Example $[-1,1]$ is a nbd of 0 in $\mathbb{R}$; it's not a nbd of -1 or 1.

**Prop 6.2** Let $(X, T_X)$ be a top space.

- $U \subseteq X$ is open (i.e. $U \in T_X$) $\implies$ $U$ is a neighborhood of every $x \in U$.

**Proof** ($\Rightarrow$) If $U$ is open, then $\forall x \in U, \ x \in U \subseteq U$. $\implies$ $U$ is a nbd of $x$.

($\Leftarrow$) Suppose $U \subseteq X$ and $\forall x \in U, \exists V_x$ open with $x \in V_x \subseteq U$.

Then $U = \bigcup V_x$ is open.

**Def 6.3** Let $(X, T_X)$ be a top space, $A \subseteq X$ a subset. A point $x \in X$ is a limit point of $A$ if every nbd $N$ of $x$ contains a point of $A \setminus \{x\}$: $\forall$ nbd $N$ of $x$, $N \cap (A \setminus \{x\}) \neq \emptyset$.

**Examples**

1. $A = [0,1] \subseteq \mathbb{R}$. Every point of $A$ is a limit point of $A$ and nothing else is: $[0,1]' = [0,1]$.  

**Notation** $A' = \text{set of limit points of } A \subseteq (X, T_X)$

2. $A = \{0,1\} \subseteq \mathbb{R}$. $A' = \emptyset$.

3. $A = (0,1) \subseteq \mathbb{R}$. $A' = [0,1]$.

Recall $C \subseteq (X, T_X)$ is closed $\iff X \setminus C \in T_X$.

**Def 6.4** Let $A$ be a subset of a top space $(X, T_X)$. The closure $\bar{A}$ of $A$ is the smallest closed subset of $X$ containing $A$:

- If $C \subseteq X$ is closed and $A \subseteq C$ then $\bar{A} \subseteq C$.  

Remark Closure $\overline{A}$ exists and is unique (so the notation $\overline{A}$ makes sense).

Reason let $\overline{A} = \bigcap_{C \in \mathcal{C}} C$. Easy to see that

1) $\overline{A}$ is closed
2) $A \subseteq \overline{A}$
3) $C \subseteq X$ closed with $A \subseteq C$ then $\overline{A} = \bigcap_{C \subseteq X \text{closed}} A \subseteq C$.

$\blacksquare$

Remark $A$ is closed $\iff A = \overline{A}$.

Proposition 6.5 Let $X$ be a top space, $A \subseteq X$. Then $\overline{A} = A \cup A'$

In particular, $x \in \overline{A} \iff \forall$ nbd $N$ of $x$, $N \cap A \neq \emptyset$.

Proof We argue $x \notin \overline{A}$ $\iff \exists$ nbd $N$ of $x$ with $N \cap A = \emptyset$.

($\Rightarrow$) If $x \notin \overline{A}$, then $x \notin X \setminus \overline{A}$. Since $\overline{A}$ is closed, $N = X \setminus \overline{A}$ is open.

$\Rightarrow$ $N$ is a nbd of $x$ with $N \cap \overline{A} = \emptyset$ (hence $N \cap A = \emptyset$).

($\Leftarrow$) Suppose $\exists$ nbd $N$ of $x$ with $N \cap A = \emptyset$. Then $\exists U \subseteq X$ open with $x \in U \subseteq N$; so $U \cap A = \emptyset$. Then $A \subseteq X \setminus U \Rightarrow C = X \setminus U$ is a closed set containing $A$ with $x \notin C$. $\Rightarrow x \notin \overline{A}$.

Note: $x \notin \overline{A} \iff \exists$ nbd $N$ of $x$ with $N \cap A = \emptyset \iff x \notin \overline{A}$ and $x \notin A'$.

Def 6.6 Let $(X, T_X)$ be a top space, $\{x_n\} \subseteq X$ a sequence (i.e., a map $n \mapsto x_n$). The sequence $\{x_n\}$ converges to $y \in X$ if $\forall$ nbd $W$ of $y \exists N \in \mathcal{N}$ so that $n \geq N \Rightarrow x_n \in W$.

We say: $y$ is a limit or $\{x_n\}$ and write $x_n \to y$.

You may remember from analysis: limits of sequences in $\mathbb{R}^n$ are unique.

$A \subseteq \mathbb{R}^n$, $x \notin \overline{A}$ $\Rightarrow \exists$ a sequence $\{x_n\} \subseteq A$ with $x_n \to y$. 
Both statements are false for arbitrary top spaces.

\(\exists x \in X = \{a, b, c\}, \text{ three point space. } \quad T_x = \{\emptyset, X, \{a, b, c\}, \{b, c\}\}.

\[X: \quad \bullet a \quad \bullet b \quad \bullet c\]

Let \(x_n = b \forall n\) (constant sequence). Then \(x_n \to b, x_n \to a\) and \(x_n \to c\).

**Proposition 6.7** \(\quad \forall x \in X, \forall n \in \mathbb{N} \text{ with } x_n \to x. \quad \exists y \in \overline{A}.

**Proof** Since \(x_n - x \notin \text{nbd } W\) of \(y\), \(\emptyset \neq W \ni 4x_n \in W \ni A. \Rightarrow y \in \overline{A}.

**Example 2** Consider \(X = \mathbb{R}^\infty = \prod_{i=1}^{\infty} \mathbb{R}_i\) with \text{box topology}.

The basis \(B_{\text{box}} = \{\prod_{i=1}^{\infty} U_i \mid U_i \subset \mathbb{R}_i \text{ open}\}.

Let \(A = \{x \in \mathbb{R}^\infty \mid x_n > 0 \forall n\}.

Let \(q = \text{constant sequence } x_n = 0 \forall n\).

If \(W = \text{any nbd of } q \text{ in } (\mathbb{R}^\infty, T_{\text{box}})\). Then \(\exists \text{ element } TU\)

of the basis \(B_{\text{box}}\) with

\[0 \in TU; \quad 0 \in U_i \ni U_i \Rightarrow \exists \varepsilon_i > 0 \ni (-\varepsilon_i, \varepsilon_i) \subset U_i.

\Rightarrow \forall i; U_i \cap (0, 0) \neq \emptyset \Rightarrow TU \cap A \neq \emptyset.

On the other hand, if \(\{x_n\}_{n \in \mathbb{N}}\) is a sequence in \(A\)

Then \(\forall n \in \mathbb{N}, \quad x_n = \frac{1}{n} x_n^{(k)}\) \text{ sequence in } A

Consider

\[U := \prod_{n=1}^{\infty} (-x_n^{(n)}, x_n^{(n)}).

Then \(x_n \notin U \forall n\), since \(x_n^{(n)} \notin (-x_n^{(n)}, x_n^{(n)})\)

\[\Rightarrow \quad \{x_n\} \notin U\]

Thus \(y \in \overline{A} \Rightarrow \exists \text{ sequence } \{a_n\} \in A \text{ with } a_n \to y.\)
Uniqueness of limits is easy to "fix":

**Def. 6.8** A topological space \((X, \mathcal{T}_X)\) is **Hausdorff** \((T_2)\) if \(\forall x, y \in X\) with \(x \neq y\) there exist open sets \(U, V\) with \(x \in U, y \in V, U \cap V = \emptyset\).

**Ex** Any metric space \((X, d)\) with metric topology \(\mathcal{T}_d\) is Hausdorff: if \(x, y \in X, x \neq y\), then \(r = d(x, y) > 0\).

\[ \Rightarrow B_{r/2}(x) \cap B_{r/2}(y) = \emptyset. \]

**Proposition 6.9** Suppose \((X, \mathcal{T}_X)\) is a Hausdorff top space, \((x_n)_{n \in \mathbb{N}} \subseteq X\) a sequence with \(x_n \to y\) and \(x_n \to z\). Then \(y = z\) (i.e., limits of sequences, if they exist, are unique).

**Proof** Suppose \(y \neq z\). Then \(\exists U, V \in \mathcal{T}_X\) sat.

\[ x_n \in U, z \in V \text{ and } U \cap V = \emptyset. \]

Since \(x_n \to y\), \(\exists N > 0\) so that \(n \geq N \Rightarrow x_n \in U\).

\[ \Rightarrow \text{for } n \geq N, x_n \notin U. \Rightarrow x_n \not\to z. \text{ Contradiction.} \]

Therefore \(y = z\). \(\square\)

There are two ways to "fix" the problem with Example 2:

- put a restriction on the topology - require \((X, \mathcal{T}_X)\) to be "1st countable"

- replace sequences by something more general - nets

This amounts to replacing integers \(n\) in the definition of a sequence by possibly bigger indexing sets.