Claim: $X$ has a simply connected covering space $p : \hat{X} \to X$.

Idea: Fix $x_0$ and consider $\text{star}(x_0) = \{ y \in \tilde{X} \mid \tau(y) = x_0 \}$. Together with $p(\tau(x)) = \tau(1) = \tau(\tau(y))$, we make $\text{star}(x_0)$ into a top space so that $p : \text{star}(x_0) \to X$ is a covering map (and $\text{star}(x_0)$ is connected).

Note: $p'(x_0) = \gamma | \tilde{X} : \tilde{X} \to X \mid \gamma(0) = \gamma(1) = x_0$ = $\pi_1(X, x_0)$ and $\pi_1(X, x_0)$ acts on $p'(x_0)$ by left multiplication: $\pi_1(X, x_0) \times p'(x_0) \to p'(x_0)$, $[\sigma] * [\gamma] = [\sigma * \gamma]$.

This action is free and transitive. So once we know $p$ is a covering map and the action comes from the lift of curves, we know $\text{star}(x_0)$ is simply connected.

Suppose $U \subseteq X$ is open, path connected and $(iu)_* : \pi_1(U, x) \to \pi_1(X, x)$ is trivial for some $x \in U$. Then $\forall x' \in U$ ad any two paths $\sigma, \tau \in U$ from $x$ to $x'$, $\sigma^{-1} * \tau$ is a loop in $U$ which is contractible in $X$, $\Rightarrow$

$[\sigma] = [\tau^{-1} * \sigma] = [\tau] \ast [\sigma] \in \tilde{X}$. 

$\Rightarrow \forall x' \in U \exists! \text{ arrow } [x'] \in \tilde{X}$ with $s([x]) = x$, $t([x]) = x'$.

$\Rightarrow \forall x' \in U$, $(iu)_* : \pi_1(U, x) \to \pi_1(X, x')$ is trivial as well. $\Rightarrow \forall U \subseteq X$ open. The inclusion $i_U : U \to X$ factors through $V \to U \to X$

$\Rightarrow \forall x \in V$ $(iv)_* : \pi(U, x) \to \pi(X, x)$ is trivial.

$\Rightarrow$ (iv) $\Rightarrow$

$\Rightarrow$ (v) $\Rightarrow$

$\Rightarrow$ (vi) $\Rightarrow$

Since $X$ is semi-locally 1-connected, $\forall x \in X$ and $\forall \text{ nbhd } U$ of $x$

If $\text{nbhd } U$ of $x$ so that $\exists \ (iv)_* : \pi(U, x) \to \pi(X, x)$ is trivial
\( (d) \) \( U \subseteq W \) and \( \forall \ U \subseteq \text{path connected} \)

Proof: Since \( X \) is semi-locally \( 1 \)-connected \( \exists \) \( \text{nbhd} \ V \) of \( x \) with \( (1.1) : \pi_1(V, x) \to \pi_1(Y, x) \) trivial. Since \( X \) is locally path connected \( \exists \text{nbhd} \ U \) of \( x \) with \( x \in U \subseteq W \cap V, \ U \subseteq \text{the desired nbhd} \).

Now consider \( \mathcal{O} = \{ U(x) \mid U \text{ open, path connected}, (1.1) : \pi_1(U, x) \to \pi_1(Y, x) \text{ trivial} \} \).

\( \mathcal{O} \) is a basis for the topology on \( X \) (c.f. \( \circ \) above).

It gives rise to a basis \( \mathcal{B} \) on \( \text{Star}(x_0) \).

\[ \forall [x] \in \text{Star}(x_0), \forall U \in \mathcal{O} \text{ with } t([x]) \in U \text{ consider} \]

\[ U_{[x]} = \left\{ [y] \times [y] \mid s([y]) = t([x]), \ t([y]) \in U \right\}. \]

By \( \mathcal{B} = \{ U([x]) \mid [x] \in \text{Star}(x_0) \} \)

Since \( t([y] \times [y_1]) = [y] \), \( \forall \ U_{[x]} : U([x]) \to U \) is a bijection.

If \( U \in \mathcal{O} \), and \( t([y]) \in \mathcal{V} \subseteq U \), then \( U_{[y]} \subseteq U_{[x]} \) and \( t(U_{[y]}) = \mathcal{V} \).

\( \Rightarrow \) Once we know \( \mathcal{B} \) is a basis, \( t(U_{[x]} : U_{[x]} \to U \) is a local homeo.

Claim: \( \forall U \in \mathcal{O}, \ U_{[x_1]} \cap U_{[x_2]} \neq \emptyset \iff U_{[x_1]} = U_{[x_2]} \).

Reason: \( [x_1] \subseteq U_{[x_1]} \Rightarrow \exists \ y \in t^{-1}(U) \cap t^{-1}(U) \text{ with } [y] = [y_1] \times [y_1]. \)

\( U_{[x_1]} \cap U_{[x_2]} \neq \emptyset \Rightarrow \exists [y], [y_1], [y_2] \in t^{-1}(U) \cap t^{-1}(U) \text{ with } [y_1] = [y_2] \times [y]. \)

Now \( [x_1] + U_{[x_2]} \Rightarrow [y_1] = [z_1] + [x_1] \), for some \( [z_1] \in t^{-1}(U) \cap t^{-1}(U) \)

\[ = ([x_1] \times [y_1])^{-1} \times [y_2] \times [y_1] \in U_{[x_1]}. \]

\( \Rightarrow U_{[x_1]} \subseteq U_{[x_2]} \) and by symmetry \( U_{[x_2]} \subseteq U_{[x_1]} \).

Remark: \( p : \text{Star}(x) \to X \quad \forall \pi(X) = t([x]). \) So \( X = \pi(X) \)

\( \mathcal{P}^{-1}(x) = \{ [y] \in \pi(X) \mid x \times [x] \times [y] \}

\( \Rightarrow \forall U \in \mathcal{O}, \mathcal{P}^{-1}(U) = \{ [x] \in \mathcal{P}(x) \mid U \subseteq U_{[x]} \}. \)
Claim \( \mathcal{B} \) is a basis for a topology on \( \text{star}(x_0) \).

**Proof** Suppose \( U \cap \overline{Y} \cap V \neq \emptyset \).

Then \( \exists \{ \eta \} \in \mathcal{B}(U) \cap \mathcal{B}(V), \{ \eta \} \in \mathcal{B}(U) \cap \mathcal{B}(V) \) so that

\[
[\mathcal{B}(U) \cap \mathcal{B}(V)] = \mathcal{B}(U) \cap \mathcal{B}(V).
\]

Let \( [\eta] = [\mathcal{B}(U) \cap \mathcal{B}(V)] \). Then \( U = U_{[\mathcal{B}(U) \cap \mathcal{B}(V)]} \) and \( V = V_{[\mathcal{B}(U) \cap \mathcal{B}(V)]} \). Let \( \eta = t([\mathcal{B}(U) \cap \mathcal{B}(V)]) \). Then \( x \in \text{U} \cap \text{V} \).

Since \( \mathcal{B} \) is a basis, \( \exists \mathcal{W} \subseteq \mathcal{B} \) with \( x \in \text{U} \cap \mathcal{V} \).

\[ \Rightarrow \quad \text{W}_{[\mathcal{B}(U) \cap \mathcal{B}(V)]} \subseteq U_{[\mathcal{B}(U) \cap \mathcal{B}(V)]} \cap V_{[\mathcal{B}(U) \cap \mathcal{B}(V)]} \]

Since \( U \cup \mathcal{B} = \text{star}(x_0), \) \( \mathcal{B} \) is a basis.

\[ \square \]

**Note:** Since \( \forall \mathcal{U} \in \mathcal{B}, \) \( p^{-1}(\mathcal{U}) = \bigsqcup_{[\mathcal{U}] \in \mathcal{B}(x)} \mathcal{U}_{[\mathcal{U}]} \) and

\[
p : \mathcal{U}(x) \to \mathcal{U} \text{ is a homeo.} \quad \forall \mathcal{U} \in \mathcal{B} \text{ is a covering map.}
\]

Claim \( \text{star}(x_0) \) (with the topology defined by \( \mathcal{B} \)) is path-connected.

**Proof** Given \( \mathcal{W} \in \text{Star}(x_0) \), consider \( F : (0,1) \times (0,1) \to X, \)

\[ F(s,t) = \gamma(st) \]

**Note** \( F(s,t) = \gamma(s), \quad \forall s \in (0,1). \)

\( F \) defines \( f : (0,1) \to \text{Star}(x_0) \) by \( f(s) = [F(s, \cdot)] = [\gamma(st)] \)

\( (p \circ f)(s) = t([\mathcal{W}(s)]) = [\gamma(st)] = \gamma(s). \)

Since \( p \circ f \) is continuous and \( p \) is a local homeo,

\( f \) is continuous. \( \Rightarrow \text{Star}(x_0) \) is path-connected.

**Conclusion** \( p : \text{Star}(x_0) \to X \) is a path-connected cover and

\( p^{-1}(x_0) \subseteq \pi_1(X, x_0). \quad \Rightarrow \quad \pi_1(\text{Star}(x_0), [1_{x_0}]) = \mathbb{Z}, \) i.e.

\( p : \text{Star}(x_0) \to X \) is a universal cover.

**Remark** \( \pi_1(X, x_0) \) acts on \( \text{Star}(x_0) \) on the left

\[
\pi_1(X, x_0) \times \text{Star}(x_0) \to \text{Star}(x_0)
\]

\[
[\mathcal{U}] \circ [\mathcal{V}] : = \mathcal{U} \times [\mathcal{V}]^{-1}
\]
For any \( \eta \neq \eta' \in \mathcal{U}_\mathcal{A} \)

\[
[\sigma] \cdot (\eta \times \eta') = (\eta \eta' \times \sigma) \times [\sigma]^{-1} - \eta' \times (\eta \times [\sigma]^{-1})
\]

\[
[\sigma] \cdot (U \times \eta) = U \times [\sigma] \cdot \eta.
\]

\[\Rightarrow \text{ action of } \pi_1(X,x_0) \text{ on } \text{Star}(x_0) \text{ is continuous.}\]

Moreover, \( \forall \sigma \in \pi_1(X,x_0), \quad \varphi[\sigma] : \text{Star}(x_0) \to \text{Star}(x_0), \)

\[
[\sigma] \cdot (\eta \times \eta') = [\sigma] \cdot (\eta \times \eta').
\]

Commuting with \( \varphi : \text{Star}(x_0) \to X. \)

\[\Rightarrow \text{ we get a map } \pi_1(X,x_0) \to \text{Hom}_{\text{Cont}}(\text{Star}(x_0),\text{Star}(x_0)).\]

\[\sigma \mapsto \varphi[\sigma] = \text{mult by } [\sigma]^{-1}.\]

Claim \( \varphi \) is an isomorphism.