Recall a continuous map \( p : Y \to X \) is a covering map if its fibers \( p^{-1}(x) \) at \( x \in X \) are discrete and \( X \times X \) is open and \( V \) and a homeomorphism \( \tilde{p} : \tilde{p}^{-1}(V) \to V \times p^{-1}(x) \) so that \( \tilde{p}^{-1}(V) \) \( \tilde{p}^{-1}(x) \) commute.

We say "\( V \) is evenly covered by \( p \)."

- We've seen that covering maps are local homeomorphisms. Not every local homeomorphism is a covering map.

**Lifting properties**

Let \( p : Y \to X \) be a map. A lift (ing) of a map \( f : Z \to X \) in a map \( \tilde{f} : Z \to Y \) so that \( \tilde{f} \circ f = f \).

A lifting need not exist: e.g. \( \exp : \mathbb{R} \to S^1 \), \( \theta \to e^{2\pi i \theta} \).

Lift of \( \text{id} : S^1 \to S^1 \) to \( \tilde{\text{id}} : S^1 \to \mathbb{R} \) with \( \exp \circ \tilde{\text{id}} = \tilde{\text{id}} \) does not exist. (meaning: \( \text{no continuous map} \tilde{\text{id}} \).

**Lemma 35.1** Let \( p : Y \to X \) be a covering map and \( f : [0,1] \to X \) a path. For any \( y_0 \in Y \) with \( p(y_0) = f(0) \) \( \exists \! \tilde{f} : [0,1] \to Y \) with \( \tilde{f}(0) = y_0 \) and \( p \circ \tilde{f} = f \).

**Proof** let \( x_0 = f(0) \), let \( U \) be an evenly covered nbhd of \( x_0 \) so that \( p^{-1}(U) = V \times p^{-1}(x_0) \). Suppose first that \( f([0,1]) \subseteq V \).

Let \( U = p^{-1}(V \times x_0) \). Since \( p|U : U \to V \) is a homeomorphism, we define \( \tilde{f} : [0,1] \to U \times Y \) by \( \tilde{f} = (p|U)^{-1} \cdot f \).

It's a desired lift of \( f \). Moreover if \( h : [0,1] \to Y \) is any other lift of \( f \) with \( h(0) = y_0 \), then \( p \circ h : [0,1] \to p^{-1}(x_0) \) is continuous, hence constant. \( \therefore \) \( f([0,1]) = U = p^{-1}(V \times y_0) \).

Since \( p \circ h = f \) and since \( p|U : U \to V \) is a homeo, we must have \( h = (p|U)^{-1} \cdot f = \tilde{f} \).
Thus if image of $f: [0,1] \to X$ lies in an evenly covered nbhd $A \ni x_0 = f(0)$, its lift $\tilde{f}$ exists and is unique.

In general, if an open cover of $f([0,1])$ by evenly covered nbhds by Lebesgue's number lemma, we can find, $0 < k < n, \, f\left(\frac{k-1}{n}, \frac{k}{n}\right)$ that lies in an evenly covered open set $V_k \in \mathcal{U}$. We construct the lift $\tilde{f}_1$ of $f$ inductively. Since $V_1 \in \mathcal{U}$ is evenly covered, there exists $\tilde{f}_1: \{0,1,2\} \to Y$ with $\tilde{f}_1(0) = y_0$, $p \circ \tilde{f}_1 = f(0,1,k)$.

Since $\frac{1}{n}, \frac{2}{n}, 1 \in \mathcal{U}$ is evenly covered, there exists $\tilde{f}_2: \{0,1,2\} \to Y$ with $\tilde{f}_2(0) = y_1$, $p \circ \tilde{f}_2 = f(0,1,k)$.

Proceeding in this way we get lifts $\tilde{f}_k: \{0,1,2\} \to Y$ of $f: \{0,1,2\} \to Y$ for all $k$. Together $\tilde{f}_0 = \tilde{f}_1 = \tilde{f}_2 = \cdots$ define a lift $\tilde{f}$ of $f$ with $\tilde{f}(0) = y_0$.

To prove uniqueness, suppose we have $g, h: [0,1] \to Y$ with $h(0) = y_0 = g(0)$ and $p \circ h = p \circ g$. Then $W = \{ t \in [0,1] \mid h(t) = g(t) \}$ is not empty.

We argue that $W$ is open and $[0,1] \setminus W$ is open. (Hence $W = \emptyset$.)

If $t \in W$, then $h(t) = g(t)$. Then $x = p(h(t))$ has an evenly covered nbhd $U \in X$. $\to p^{-1}(U) \to U \times p^{-1}(x)$ is a homeomorphism.

Let $\tilde{G} = G^{-1}(U \times p^{-1}(x))$. Since $h, g$ are continuous, for all $t \in \mathcal{U}$.

Since $p|\tilde{G}: \tilde{G} \to U$ is a homeomorphism, we must have $p(h(t)) = (p|\tilde{G})^{-1} \circ h(t) = (p|\tilde{G})^{-1} \circ g(t) = (p|\tilde{G})^{-1} \circ g(t)

for all $t \in (t-\varepsilon, t+\varepsilon) \subseteq W$.

If $t \not\in W$, then $h(t) \neq g(t)$. Let $U$ be the evenly covered nbhd of $x = p(h(t)) = p(g(t))$, and $\psi: p^{-1}(U) \to U \times p^{-1}(x)$ be a homeo.

Since $h(t) \neq g(t)$, then $\psi^{-1}(U \times \{h(t)\}) \cap \psi^{-1}(U \times \{g(t)\}) = \emptyset$.

Since $t \in (t-\varepsilon, t+\varepsilon)$.

$g((t-\varepsilon, t+\varepsilon)) \subseteq \psi^{-1}(U \times \{g(t)\})$.

$(t-\varepsilon, t+\varepsilon) \subseteq [0,1] \setminus W$.

Since $[0,1]$ is connected, $[0,1] = W \cup \{ t \in [0,1] \mid h(t) = g(t) \}$. [Diagram]

\[\rightarrow \quad \text{[Diagram]}\]
Lemma 35.2 (Homotopies of paths rel end points lift)

Suppose \( p : Y \to X \) is a covering map, \( f_0, f_1 : [0,1] \to X \) two paths with \( f_0 \sim f_1 \), rel \( \{0\} \), and \( \tilde{f}_0 : [0,1] \to Y \) in a lift of \( f_0 \). Then

\( \exists \) lift \( \tilde{F} : [0,1]^2 \to Y \) of \( F \) with \( \tilde{F} \mid (0,1) \times \{0\} = \tilde{f}_0 \).

**Proof** let \( x_0 = F(0,0) = f_0(0), \tilde{x}_0 = \tilde{f}_0(0) \). Suppose \( F([0,1]^2) \) lies in an evenly covered \( \text{nbhd} \) \( V \) of \( x_0 \). Then, as before \( p'(V) \xrightarrow{u} U \times p'(x_0) \).

Let \( U = p(V \times \{x_0\}) \) and set \( \tilde{F} = (pU)^{-1} \circ F \). Then \( \tilde{F} \) in a lift of \( F \) with \( \tilde{F}(0,0) = \tilde{x}_0 \). In particular \( \tilde{F} \mid (0,1) \times \{0\} \) is a lift of \( F \mid (0,1) \times \{0\} = f_0 \)

\( \exists \tilde{F} \mid (0,1) \times \{0\} = \tilde{f}_0 \).

*Easy to see* if \( H \) is another lift of \( F \) with \( H(0,0) = x_0 = F(0,0) \), then \( H = (pU)^{-1} \circ F = \tilde{F} \).

In general \( \exists n \in \mathbb{N} \) at \( V, U, \tilde{V} \subseteq W, 0 \leq k, l < n \),

\( F(\frac{[k, k+1]}{[0, n]} \times \frac{l, l+1}{[0, n]}) \) lies in an evenly covered open set in \( X \). The lift \( \tilde{F} \) is then constructed inductively.

**First** we define \( \tilde{F}_{0,0} : \left[ \frac{0}{n}, \frac{1}{n} \right]^2 \to Y \) so that \( \tilde{F}_{0,0}(0,0) = x_0 \), \( p \circ \tilde{F}_{0,0} = F \mid \left[ \frac{0}{n}, \frac{1}{n} \right]^2 \).

Then define

\( \tilde{F}_{0,0} : \left[ \frac{k}{n}, \frac{k+1}{n} \right] \times \left[ \frac{l}{n}, \frac{l+1}{n} \right] \to Y \) with

\[ p \circ \tilde{F}_{0,0} = F \mid \left[ \frac{k}{n}, \frac{k+1}{n} \right] \times \left[ \frac{l}{n}, \frac{l+1}{n} \right] = \tilde{F} \mid \left[ \frac{k}{n}, \frac{k+1}{n} \right] \times \left[ \frac{l}{n}, \frac{l+1}{n} \right] \]

Proceeding inductively and gluing these \( \tilde{F}_{k,0} \) together gives us \( \tilde{F} \mid \left[ 0,1 \right] \times \left[ 0, \frac{1}{n} \right] \). Next define \( \tilde{F}_{1,0} : \left[ 0, \frac{1}{n} \right] \times \left[ \frac{1}{n}, \frac{2}{n} \right] \to Y \), etc.

This gives existence of a lift \( \tilde{F} \) of \( F \).

Since \( \tilde{F} \mid \left[ 0,1 \right] \times \{0\} \) is a lift of \( F \mid \left[ 0,1 \right] \times \{0\} = f_0 \)

with \( \tilde{F}(0,0) = \tilde{x}_0 \), \( \tilde{F} \mid \left[ 0,1 \right] \times \{0\} = \tilde{f}_0 \).

Uniqueness is proved as in the previous lemma.
Remark: By construction

\[ \bar{F}(0,t) = F(0,t) = x_0 \quad \forall t \in [0,1] \]

\[ \Rightarrow \bar{F}(0,t) \in p^{-1}(x_0) \text{, which is discrete.} \]

\[ \Rightarrow \bar{F}(0,t) = F(0,0) = y_0 \quad \forall t. \]

Similarly, \[ \bar{F}(1,t) = F(1,0) \quad \forall t \]

\[ \Rightarrow \bar{F} \text{ defines a homotopy from } \bar{f}_0 \text{ to } \bar{f}_1 \text{ rel } 10,15, \text{ where} \]

\[ \bar{f}_1 \text{ is the lift of } f_1 \text{ with } \bar{f}_1(0) = y_0. \]

Moreover, if \[ x_0 \xrightarrow{\tilde{g}} x_1 \] are two paths in \( X \)
and \( y_0 \in Y \) as before, \( p(y_0) = x_1 \), then \( (\tilde{g} \circ \tilde{f})_{y_0} \) is a lift
of the concatenation of \( \tilde{f} \) and \( \tilde{g} \) with \( (\tilde{g} \circ \tilde{f})_{y_0}(0) = y_0 \).

Uniqueness of lifts:

\[ \tilde{g} \circ \tilde{f}_{y_0} = (\tilde{g} \circ p)(\tilde{f}_{y_0}), \quad \text{re} \]

\[ \tilde{g} \circ \tilde{f} \text{ is the concatenation of the lift } \tilde{f}_{y_0} \text{ of } \tilde{f} \text{ with } \tilde{f}_{y_0}(0) = y_0 \]
followed by the lift of \( g \) with \( \tilde{g}(0) = \tilde{f}(1) \).

**Lemma 3.2**

There is an arrow \( x_0 \xrightarrow{\tilde{f}} x_1 \) in \( \Pi_1 X \), and \( y_0 \in p^{-1}(x_0) \)
we get a unique arrow \( y_0 \xrightarrow{\tilde{g}} y_1 \) in \( \Pi_1 Y \), where \( y_1 \in p^{-1}(x_1) \)
and \( \tilde{g} \) depends only on \( \tilde{f} \) and \( y_0 \).