Covering spaces

Suppose \( f: Y \to X \) is continuous. The fiber of \( f \) at \( x_0 \in X \)
in the space \( f^{-1}(x_0) \subset Y \)

\[ f: Y \to X \]
is a covering map (and \( Y \) is a covering space of \( X \)) if:
1. \( \forall x \in X \) the fiber \( f^{-1}(x) \) is discrete and
2. \( \forall x \in X \) \( \exists \) open \( U \) of \( x \) and a homeomorphism
   \[ \psi: f^{-1}(U) \to U \times f^{-1}(x) \]

So that \( f^{-1}(U) \xrightarrow{\psi} U \times f^{-1}(x) \)

\[ \xrightarrow{\psi} \]

\[ U \]

commutes.

\[ \text{Ex} \]
\[ X \text{ any space, } Y = X \times \{0,1\}, f(x,i) = x. \]

is a covering map of space.

\[ \text{Ex} \]
\[ \exp: IR \to S^1 = \{x \in C | |x| = 1\}, \exp(t) = e^{2\pi it} \]
in a covering map.

\[ \text{Ex} \]
\[ \exp: C \to C \times C, \quad x+iy \mapsto e^{x+iy} \]
in a covering map.

\[ \text{No Ex} \]
\[ f: (0,1) \to S^1, \quad t \mapsto e^{2\pi it} \text{ is not} \]
a covering map.
Non Ex \[ \tilde{f} : (0, 1.5) \to S^1, \quad t \mapsto e^{2\pi it} \]

\[ \text{is not a covering map} \]

We'll study covering spaces/maps for a nice class of spaces.

**Def.** A space \( X \) is semi-locally simply connected if \( \forall x \in X \)

\[ \exists \text{ a nbd } U \text{ of } x \text{ so that the map} \]

\[ \pi_1(U, x) \xrightarrow{\text{incl}} \pi_1(X, x) \]

is trivial.

i.e., any loop at \( x \) in \( U \) is contractible in \( X \).

**Ex.** Any space locally homeomorphic to \( B^n(1) = \{ x \in \mathbb{R}^n \mid ||x|| < 1 \} \) for some \( n \) is semi-locally simply connected.

So all manifolds are OK.

Non Ex \[ X = \bigcup_{n=1}^{\infty} \{ x \in \mathbb{R}^2 \mid ||x-(n,0)|| = \frac{1}{n} \} \subset \mathbb{R}^2 \]

\[ \text{is not semi locally simply connected: any nbd of } (0,0) \text{ contains a non-contractible loop.} \]

**Ev.** \( X \) as above; \( CX = X \times [0,1] / (x,1) \sim (x',1) \quad (x,x') \in X \times X \)

\[ \text{is semi locally simply connected.} \]
"Recall" A space $X$ is locally path connected if $\forall x \in X$ and any nbd $W$ of $x$, $\exists$ nbd $U$ of $x$ with $x \in U \subseteq W$ and $U$ is path connected.

We'll study locally path connected semi-locally 1-connected spaces.

Def: A continuous map $f: Y \to X$ is a local homeomorphism if $\forall y \in Y$ $\exists$ nbds $U$ of $y$ in $Y$, $V$ of $f(y)$ in $X$ st. $f|_U: U \to V$ is a homeomorphism.

Easy to see: (1) covering maps are local homeomorphisms.
(2) local homeomorphisms are open maps.
(3) a covering map is a quotient map.
(4) if $\zeta: Z \to Y$ and $\eta: Y \to X$ are covering maps then $\zeta \circ \eta: Z \to X$.

Given a space $X$ we have a collection of covering spaces $\forall Y \to X$ of $X$. It can be made into a category $\text{Conv}_X$ as follows:

\[ \text{Hom}_{\text{Conv}_X}(Y \to X, Z \to X) = \{ h: Y \to Z | h \circ f = g \text{ commutes} \} \]

Composition:
\[
\left( \begin{array}{c}
W \leftarrow^k Z \\
Y \\
X \\
\end{array} \right) \circ \left( \begin{array}{c}
Z \leftarrow^h Y \\
Y \\
X \\
\end{array} \right) = \left( \begin{array}{c}
W \leftarrow^{k \circ h} Y \\
Y \\
X \\
\end{array} \right)
\]
**WARNING** In studying covering spaces \( Y \to X \), one usually assumes: \( Y \) and \( X \) are path connected.

One then proves: if \( X \) is path connected, locally path connected, and semi-locally path connected, then there is a bijection

\[
\{ \text{no classes of covering spaces of } X \} \to \{ \text{iso classes of subgroups of } \pi_1(X, x) \} \]

\[
\{ Y \to X \} \leftrightarrow L\pi_1(\pi_1(Y, y))
\]

Back to work: fiber products.

Let \( \begin{array}{c} X \to \mathbb{Z} \\ \downarrow g \to y \end{array} \) be a diagram in a category \( C \).

A fiber product \( X \times_Y Z \) is an object of \( C \) together with two arrows \( X \leftarrow X \times_Y Z \to Z \) so that

\[
x \times_Y Z \xrightarrow{pr_2} y
\]

\[
pr_1 \downarrow \quad \downarrow g
\]

\[
X \xrightarrow{f} \mathbb{Z} \quad \text{commutes.}
\]

This is like the definition of pushout, but we reversed all the arrows.

**Prop 34.1** A diagram \( X \to Z \to Y \) has the fiber product if \( Y \) is a topological space.

**Proof**

Let \( X \times_Y Z = \{ (x, y) \in X \times Y \mid f(x) = g(y) \} \), a subspace of \( X \times Y \).

Let \( pr_1 = \pi_1|_{X \times_Y Z} \), where \( \pi_1 \colon X \times Y \to X \),

\( pr_2 = \pi_2|_{X \times_Y Z} \), where \( \pi_2 \colon X \times Y \to Y \).