Our goal is to understand the fundamental groupoid $\Pi X$ in the pushout of the diagram $\Pi U_0 \Leftarrow \Pi U_0 \sqcup U_1 \rightarrow \Pi U_1$ in the category of groupoids.

As a warm-up we are trying to understand pushouts in general. We haven't quite finished proving Claim pushouts ("amalgamated free products") exist in the category of groups.

I have sketched the existence of the pushout $G_i \xleftarrow{114} G_0$, called the free product $G_i \times G_0$.

It is (essentially) a set of words of the form $x_1 y_1 \ldots x_n y_n$, $x_i \in G_i$, $y_i \in G_0$, $n \geq 0$, with multiplication = concatenation.

The pushout $G_i \xleftarrow{H} G_0$, called, the amalgamated free product, and denoted by $G_i \times_H G_0$ in the quotient $G_i \times G_0$ by the normal subgroup generated by $\beta(h)\beta(h)^{-1}$ for $h \in H$.

Example $G_i = \mathbb{Z} \times \mathbb{Z} = \langle a, b \mid \text{no relations} \rangle$

$G_0 = \mathbb{Z} \times \mathbb{Z}$

$H = \mathbb{Z} = \langle c \rangle$

$\alpha(c) = a b a^{-1} b^{-1}$

(so $\alpha(c^n) = (a b a^{-1} b^{-1})^n$)

$\beta(c) = 1$ (no choice here)

$G_i \times_H G_0 = \langle a, b \mid \text{no relations} \rangle \langle a b a^{-1} b^{-1} = 1 \rangle$

$= \langle a, b \mid a b a^{-1} b^{-1} = 1 \rangle = \langle a, b \mid a b = b a \rangle$

$\cong \mathbb{Z} \oplus \mathbb{Z}$
\[ G_1 = \mathbb{Z} \times \mathbb{Z} = \langle a \rangle \times \langle b \rangle = \langle a, b \mid \text{no relations} \rangle \]
\[ G_0 = \mathbb{Z} \quad H = \mathbb{Z} = \langle c \rangle \quad \alpha(c) = abab^{-1}, \beta(c) = 1. \]
\[ G_1 \ast H G_0 = \langle a, b \mid a, b, a^{-1} = 1 \rangle
\[ = \langle a, b \mid b, a^{-1} = a^{-1} \rangle \]

We'll see later that \( G_1 \) computes \( \pi_1(S^1 \times S^1) \) and \( G_2 \) computes \( \pi_1 \) of the Klein bottle \( \mathbb{R} \mathbb{H} \).

We next use Brown-Seifert-van Kampen to compute \( \pi_1 \).

Claim: \( \pi_1 \mathbb{S}^1 \times \mathbb{R} \cong \mathbb{S}^1 \times \mathbb{R} \) where \( s(x, \theta) = x, t(x, \theta) = x e^{2\pi i \theta} \).

We think of \( \mathbb{S}^1 \) as \( s^1 = 1, \lambda \in \mathbb{C} \mid |1\lambda| = 1 \).

Multiplication is given by \( (x e^{2\pi i \theta}, \theta')(x, \theta) = (x, \theta + \theta') \).

Proof: Let \( U_0 = \mathbb{S}^1 \times \{ 0 \}, \quad U_1 = \{ 1 \} \times \mathbb{S}^1 \)  \( U_0 \cap U_1 = U_{01} \cup U_{01} \).

Since \( U_0 = (0, 2\pi), \quad \pi U_0 \cong \mathbb{U} \times U_0 _0 \cong \mathbb{U} \) \( s(x, x') = x, \quad t(x, x') = x' (x', x') = (x, x'') \).

Similarly, \( \pi U_1 = \mathbb{U} \times U_1 = \mathbb{U} \times \mathbb{U} \).

We have two functors \( \pi_k \mathbb{S}^1 \mathbb{R} \rightarrow \pi_1 \mathbb{S}^1 \mathbb{R} \).

\( \pi_k \mathbb{S}^1 \mathbb{R} \rightarrow \pi_1 \mathbb{S}^1 \mathbb{R} \) making

We also have a functor \( g: \{ \mathbb{S}^1 \mathbb{R} \rightarrow \mathbb{R} \} \rightarrow \pi_1 \mathbb{S}^1 \mathbb{R} \)

\( g(x, \theta) = (x, \frac{1}{2\pi} \log(\frac{\mu}{\lambda})) \) where \( \pi_1 \mathbb{S}^1 \mathbb{R} = \mathbb{R} \mathbb{S}^1 \mathbb{R} \) commute.
We need to check: \( f \circ g \) are inverses if each other. 30.3

Call a path \( \gamma \) in \( S^1 \) short if its image lies entirely in \( U_0 \) or \( U_1 \).

Similarly an arrow \((\lambda, \theta) \) in \( S^1 \times [0,1] \) is short if \( 0 \leq \theta < 2\pi \).

On short arrows \( f \circ g \) are inverses of each other.

So they are inverses if each other on all arrows.

**Proof of B-S-vK.** We need a number of observations.

1. A continuous map \( \gamma : [a,b] \to W \) represents an arrow \( \gamma(a) \overset{\lambda}{\to} \gamma(b) \) in \( \Pi W \), where

\[
[\gamma] = \text{homotopy class of } \gamma : [0,1] \to [a,b] \overset{\gamma}{\to} W, \quad s \mapsto \gamma(a + (b-a)s)
\]

2. If \( a = t_0 < t_1 < \cdots < t_n = b \) is a partition of \( [a,b] \), \( \gamma : [a,b] \to W \)

as above and \( \gamma' = \gamma|_{[t_{i-1},t_i]} \) then

\[
[\gamma] = [\gamma_0] \cdots [\gamma_n] \quad \text{(multiplication in \( \Pi W \))}
\]

3. Lebesgue lemma :

(a) If \( \{U_k\} \) is an open cover of a space \( W \) and

\( \gamma : [0,1] \to W \) is a path, then \( \exists n \in \mathbb{N} \) so that \( \forall k \)

\( \gamma \left( \left[ \frac{k}{n}, \frac{k+1}{n} \right] \right) \subseteq U_{\lambda_k} \) for some \( \lambda_k \).

(b) If \( H : [0,1] \times [0,1] \to W \) is a homotopy, \( \exists n \) st.

\( H \left( \left[ \frac{k}{n}, \frac{k+1}{n} \right] \times \left[ \frac{l}{n}, \frac{l+1}{n} \right] \right) \subseteq U_{\lambda_{k,l}} \) for some \( \lambda_{k,l} \) and all \( 0 \leq k, l < n \).

Now given a groupoid \( \Gamma \) and two functors \( f_k : \Pi U_k \to \Gamma, k = 0 \),

\[\begin{array}{ccc}
\Pi U_0 & \xrightarrow{f_0} & \Gamma \\
\Pi U_1 & \xrightarrow{f_1} & \Gamma
\end{array}\]

we construct \( f : \Pi X \to \Gamma \) as follows:
If an arrow \( \gamma : [0,1] \to X \) with \( \gamma([0,1]) \subseteq U_k \), \( k=1,2 \), we're forced to define

\[ f(L_\gamma) = f_k(L_\gamma). \]

Note: if \( \gamma([0,1]) \subseteq U_0 \cap U_1 = U_{01} \), then

\[ f_0 \circ \pi_0 (L_\gamma) = f_1 \circ (\pi_1 (L_\gamma)) \]

So there is hope that \( f \) is well-defined on "short" paths, since \( f(L_\gamma) \) so

If \( \gamma : [0,1] \to X \) is an arbitrary path \( \in \Gamma \) such that \( A_k \)

\[ \gamma(L_{\frac{k}{m}}, \frac{k+1}{m}) \subseteq U_{0k} \]

for some \( \alpha_k \)

Then \( L_\gamma = L_{\gamma_1} \cdots L_{\gamma_l} \) with \( \gamma_k \in \gamma \left[ \frac{k}{m}, \frac{k+1}{m} \right] \)

and we set

\[ (\star) \quad f(L_\gamma) = f_{\alpha_1}(L_{\gamma_1}) \cdots f_{\alpha_l}(L_{\gamma_l}). \]

Does \( f(L_\gamma) \) depend on the choice of a partition \( \Pi \) \([0,1]\)?

If \( 0 = \frac{1}{m} < \frac{2}{m} \cdots < \frac{m}{m} = 1 \) is another partition, we can take a common refinement. Then \( \gamma = \gamma \left[ \frac{k}{m}, \frac{k+1}{m} \right] \)

is replaced by concatenations of paths

\[ \gamma_k^l = \gamma \left[ \frac{km + l}{mn}, \frac{km + l + 1}{mn} \right], \]

for \( l \leq m \)

Since each \( f_{\alpha_k} \) is a functor,

\[ f_{\alpha_k}(L_{\gamma_k}) = f_{\alpha_k}(L_{\gamma_k^l}) \cdots f_{\alpha_k}(L_{\gamma_k^m}) \]

\( \Rightarrow \)

\( f(L_\gamma) \) doesn't depend on the choice of a partition.

Finally we need to check that \( \gamma \approx_{\mathcal{H}} \gamma' \) rel \([0,1]\), then

\[ f(L_\gamma) = f(L_\gamma'). \]