Recall that $F : B \to D$ is part of an equivalence of categories if $F \circ G : D \to B$ is a functor, and natural isomorphisms
\[ \alpha : G \circ F \Rightarrow \text{id}_B \quad \beta : F \circ G \Rightarrow \text{id}_D, \]
are available.

(2) $F : B \to D$ is fully faithful if for any $c, c' \in B$,
\[ F : \text{Hom}_B (c, c') \to \text{Hom}_D (F(c), F(c')) \]

is a bijection.

$F : B \to D$ is essentially surjective if any $d \in D_0$ is isomorphic to $F(c)$ for some $c$. Then $d \xrightarrow{\alpha_d} F(c)$ for $c$.

27.2 $F : B \to D$ is part of an equivalence of categories if $F$ is fully faithful and essentially surjective.

Proof: Suppose $F : B \to D$, $\alpha : G \circ F \Rightarrow \text{id}_B$, and $\beta : F \circ G \Rightarrow \text{id}_D$. Then any arrow $c \to c'$ in $B$, the diagram
\[ \begin{array}{ccc}
GF(c) & \xrightarrow{\alpha} & c \\
\downarrow f & \Downarrow & \downarrow \beta \\
GF(c') & \xrightarrow{\alpha} & c'
\end{array} \]

commutes, with $\alpha_c, \alpha_{c'}$ isomorphisms.

If $c \xrightarrow{f} c'$ are two arrows with $F(f) = F(f')$, then
\[ GF(f) = GF(f') \]

\[ \Rightarrow f = \alpha_{c'} \circ GF(f) \circ \alpha_c^{-1} = \alpha_{c'} \circ GF(f') \circ \alpha_c^{-1} = f'. \]

$F : \text{Hom}_B (c, c') \to \text{Hom}_D (F(c), F(c'))$ is $1-1$, so $F$ is faithful.

Similarly, $G : \text{Hom}_D (d, d') \to \text{Hom}_B (G(d), G(d'))$ is $1-1$.

We now argue that $F : \text{Hom}_B (c, c') \to \text{Hom}_D (F(c), F(c'))$ is onto. Say $h : \text{Hom}_D (F(c), F(c'))$. Then $G(h) \in \text{Hom}_B (GF(c), GF(c'))$.

Let $f = \alpha_{c'} \circ G(h) \circ \alpha_c^{-1}$. Then $\begin{array}{ccc}
f & \downarrow \alpha_c & GF(c) \\
\downarrow f & \Rightarrow & \downarrow \alpha_{c'} \\
c' & \xleftarrow{\alpha_c} & GF(c') \end{array}$

commutes.

By definition of $\alpha$,
Given \( d \in D \) we have \( \beta_d : F(G(d)) \rightarrow d \), an iso.

Hence \( c = G(d) \) is an element of \( C \) with \( F(c) \xrightarrow{\beta_d} d \).

\[ \Rightarrow \text{F is essentially surjective.} \]

\( \Leftarrow \) Suppose \( F : C \rightarrow D \) is fully faithful and essentially surjective. We want to define a functor \( G : D \rightarrow C \) and two natural transformations \( \alpha : GF \Rightarrow \text{id}_C \)

\[ \beta : FG \Rightarrow \text{id}_D. \]

Since \( F \) is essentially surjective \( \forall d \in D \) \( F(c) \) is an iso and an iso \( F(c) \xrightarrow{\beta_d} d \) in \( D \). Choose one such \( c \) and \( \beta_d \) for each \( d \).

Define \( G : D \rightarrow C \) by \( G(d) = c \).

Given \( d \rightarrow d' \) in \( D \) we have \( d \xrightarrow{\beta_d} F(G(d)) \)

\[ \beta_d' \]

\[ F(G(d')) \xrightarrow{\beta_d'} d' \]

\[ \Rightarrow \text{F is a bijection.} \]

Define \( G(h) = F^{-1}(\beta_{d'}^{-1}h \circ \beta_d) \).

\[ \text{Note: } G(\text{id}_d) = F^{-1}(\beta_{d'}^{-1}\beta_d) = F^{-1}(\text{id}_C) = \text{id}_C \]

\[ F(G(d)) \xrightarrow{\beta_d} d \]

\[ F(G(d')) \xrightarrow{\beta_d'} d' \]

\[ \text{Commutates.} \]

It's not hard to check \( G \) preserves composition.
To find $\alpha : GF \Rightarrow \text{id}_G$ (i.e., to find $\alpha_G : GF(G(c)) \to c \quad \forall c \in G_0$), consider $\beta_{F(c)} : FGF(c) \cong F(c)$.

Since $F : \text{Hom}_G(GF(c), c) \to \text{Hom}_D(FGF(c), F(c))$ is a bijection, we define $\alpha_G = F^{-1}(\beta_{F(c)}) : GF(c) \to c$.

It is not hard to check that $\forall c \in c'$

$$GF(c) \xrightarrow{F^{-1}(\beta_{F(c)})} c$$

commutes. □

We use 27.2 to prove: if $\phi : X \to Y$ is a homotopy equivalence, then $\forall x \in X$$

$$\Pi \phi : \Pi_1(X, x) \to \Pi_1(Y, y(x))$$

is an isomorphism of groups.

But we haven't computed any fundamental groupoids except for convex subsets of $1R^1$.

We could try to compute $\Pi S^1$ as follows: cover $S^1 \subseteq C$ by $U_0 = S^1 \setminus \{y\}$ and $U_1 = S^1 \setminus \{-y\}$.

Let $U_0 \downarrow = U_0 \cap U_1$. We have a commuting diagram of spaces (in $\text{Top}$)

So we have a commuting diagram of groupoids (in $\text{Grpd}$)

$\Pi U_0 \xrightarrow{\Pi \phi_0} \Pi U_1$

$\Pi U_0 \to \Pi S^1$
Universal property of $S^1$; given $f_0: U_0 \to X$, $f_1: U_1 \to X$ so that $f_0 \circ i_0 = f_1 \circ i_0$. Thus $f: S^1 \to X$ with $f \circ i = f_i$ for $i = 0, 1$.

Wilde guess/hope: diagram 2 has the same universal property. It is true. It's Brown-Seifert-vonKampen Theorem.

**Def:** Let $A$ be a category $a_0, a_1, a_1, e \in A$, and $i_1: a_0 \to a_1$, $i_0: a_0 \to a_1$ two morphisms.

(We say $i_0 \dashv a_0 \to a_1$ in a diagram in $A$.)

An object $x \in A$ and a pair of arrows $a_0 \to x \to a_1$ in a pushout of (4) if a pair of arrows $f_0: a_0 \to z$, $f_1: a_1 \to z$ so that $i_0 \dashv f_0$ and $i_1 \dashv f_1$ commute (i.e. $f_1 \circ i_1 = f_0 \circ i_0$).

\[ f: x \to z \text{ so that } \]

\[ f_0 \downarrow \quad f_1 \]