Last time: - homotopies of (continuous) maps,
- homotopy classes of maps. Notation \([f] = \text{homotopy class of } f\).

- definition of a category, \(h\text{Top}\)

More examples of categories:

"LCH" = locally compact Hausdorff spaces and proper maps
- \(\text{Top}_\ast\) = pointed spaces
  - objects are pairs \((X, x), \ x \in X\)
  - morphisms \(f : (X, x) \to (Y, y), \ f : X \to Y\) contin., \(f(x) = y\)
- \(\text{Vect}_\mathbb{R}\) = category of vector spaces \(\mathbb{IR}\) and linear maps
- \( FD\text{Vect}_\mathbb{R}\) = the category of finite dim. vector spaces \(\mathbb{IR}\) and linear maps.

\(\&\) Any group \(G\) defines a category \(G\): \(G\) has one object \((e)\)
\(\text{Hom}_G(x, x) = G\) with composition \(\circ = \text{group mult.}\)
- \(1_x = \text{identity element of } G\).

**Def.** A morphism \(f : X \to Y\) in a category \(\mathcal{C}\) is an *isomorphism*
- if \(f \circ g : Y \to X\) s.t. \(f \circ g = \text{id}_X, \ g \circ f = \text{id}_Y, \)

\(\&\) If \(\mathcal{C}\) = set, then isomorphisms are bijections
- If \(\mathcal{C} = \text{Top}\) then isomorphisms are homeomorphisms
- If \(\mathcal{C} = h\text{Top}\) (top spaces and homotopy classes of maps)
  - isomorphisms are called "homotopy equivalences."

Actually, if \(f, g : X \to Y\) in \(h\text{Top}\)
- \(f\) is called a homotopy equiv.

Unpacking the \(\text{Def}\) we see: \(f : X \to Y\) is a homotopy equiv
- \(\exists g : Y \to X\) and homotopies \(g \circ f = \text{id}_X\)
- \(f \circ g \approx \text{id}_Y\).
\[ X = \ast \] 1 point space, \( Y = \mathbb{R}^n \)

\[ f: X \to Y, \quad f(\ast) = 0 \] is a homotopy equivalence.

Consider \( g: \mathbb{R}^n \to X, \ g(x) = \ast + \mathbf{x}. \)

Then \( g \circ f = \text{id}_X, \quad f \circ g(\mathbf{x}) = 0 + \mathbf{x} \)

\[ t \circ g \equiv \text{id}_{\mathbb{R}^n}, \text{ where } \]

\[ F(\mathbf{x}, t) = t \mathbf{x}. \]

It's hard to prove using bare hands that \( S^1 \) is not homotopy equivalent to a point. We'll do it by attaching to spaces categories — the fundamental groupoid. For \( X = \ast, \pi_1 = \text{trivial} \)

For \( X = S^1, \pi_1 S^1 \) is not trivial.

The following concept will be useful:

**Definition:** A functor \( F: \mathcal{C} \to \mathcal{D} \) from a category \( \mathcal{C} \) to a category \( \mathcal{D} \) is a pair of maps \( F_0: \mathcal{C}_0 \to \mathcal{D}_0, \ F_1: \mathcal{C}_1 \to \mathcal{D}_1 \) with

1. \( F_1(1_A) = 1_{F_0(A)} \) \quad \forall A \in \mathcal{C}_0 \\
2. \( F_1(g \circ f) = F_1(g) \circ F_1(f) \) for any pair of composable arrows \( g \circ f \) in \( \mathcal{C} \).

From now on we'll write \( F \) instead of \( F_0 \) or \( F_1 \).

**Example:** \((\text{LCH, proper maps}) \to (\text{Compact Hausdorff, continuos maps}) \)

**In a functor**

**Example:** \( U: \text{Top} \to \text{Set} \)

\[ U((X, \mathcal{T}_X)) = f: (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y) \]

is a functor that forgets the topology.

"Forgetful functor", "Underlying set functor"

**Example:** \( \text{Ind}: \text{Set} \to \text{Top} \)

\[ \text{Ind}(X) = (X, \{\ast, X\}) \]

\( X \) with "indiscrte" topology.

Then any \( f: X \to Y \) defines a contin map \( f: \text{Ind}(X) \to \text{Ind}(Y) \).
Def (relative homotopy). Let \( f_0, f_1 : W \to X \) be two continuous maps, \( A \subset W \) a subspace, \( f_0 \) in homotopic to \( f_1 \) relative to \( A \) if \( \exists F : W \times I \to X \) with \( F(-, 0) = f_0 \), \( F(-, 1) = f_1 \) and 
\[
F(a, t) = f_0(a) = f_1(a) \quad \forall a \in A, t \in [0, 1].
\]
(So I should have said first that \( f_0|_A = f_1|_A \).)

Special case: \( W = [0, 1] \), \( A = \{0, 1\} \) Then \( f_0, f_1 : W \to X \) are paths \( \forall t \in [0, 1] \).

Exercise \( f \simeq g \text{ rel } \{0, 1\} \) is an equivalence relation on the set of paths in \( X \).

Def. A category \( C \) is a groupoid if any morphism in \( C \) is an isomorphism.

\( \text{Ex} \) (\( C_0 = \text{finite sets} \), \( C_1 = \text{bijections} \)) is a groupoid.

\( \text{Ex} \) If \( C \) is a group, then \( C \) is a groupoid.

\( \text{Ex} \) If \( C, H \) are groups, \( C \sqcup H \) (a category with two elements \( *_C, *_H \)) \( \text{Hom}(*_C, *_C) = \emptyset \) \( \text{Hom}(*_C, *_H) = C \) \( \text{Hom}(*_H, *_H) = H \)

We now associate a groupoid \( \Pi X \) to a topological space \( X \), the fundamental groupoid.

\[
(\Pi X)_0 := X
\]
\[
\text{Hom}_{\Pi X}(x, y) = \text{homotopy classes of paths from } x \text{ to } y \text{ rel end points.}
\]
Composition = concatenation, of (homotopy classes of) paths. 24.4

Recall: concatenation is defined by:

\[ \frac{z}{z} \circ f \quad \text{if} \quad z \prec y \prec f \prec x \quad \text{are paths in } X \]

then \[ g \circ f (s) := \begin{cases} g(2s), & 0 \leq s \leq \frac{1}{2} \\ g(2s-1), & \frac{1}{2} \leq s \leq 1 \end{cases} \]

Lemma 24.1: Suppose \( g_0, g_1, f_0, f_1 \) are paths in \( X \) with \( g_0 = g_1 \), relic 0, 11, \( f_0 = f_1 \), relic 0, 11, and \( g_1(0) = f_1(1) \).

Then \( g_0 \circ f_0 = g_1 \circ f_1 \), relic 0, 11.

Proof:

\[
\begin{aligned}
\text{Define } & \quad GF(s, t) = \begin{cases} F(2s, t), & 0 \leq s \leq \frac{1}{2} \\ G(2s-1, t), & \frac{1}{2} \leq s \leq 1 \end{cases} \\
\text{For every } x \in X \text{ we set } & \quad 1_x \in \text{Hom}_{\pi(X)}(x, x) \text{ to be } (\text{the class of }) \text{ the constant path } 1_x(s) = x, \forall s \in [0, 1].
\end{aligned}
\]

To prove that \( \pi(X) \) is a category, we need to prove:

1) A triple of composable paths.

\[
( \af, \bf, \gf ) \Rightarrow LF = LLF = [x] \]

2) A \( \af \)

\[
[1_y] \circ [f] = [f] = [1_x] \]

To prove that \( \pi(X) \) is a groupoid, we need to prove:

\[
[1_y] \circ [f] = [1_x] \quad \text{and} \quad [f] \circ [g] = [1_g] \]

\[
[1_g] \circ [f] = [1_x] \quad \text{and} \quad [f] \circ [g] = [1_g].
\]