Recall

A topology \( T \) on a set \( X \) is a collection of subsets of \( X \) such that

1) \( X, \emptyset \in T \)
2) \( U, V \in T \Rightarrow U \cup V \in T \)
3) \( \forall A \subseteq X \Rightarrow U \cup A \in T \)

Elements of \( T \) are "open sets".

Ex 1f \((X,d)\) is a metric space then

\[ T_d = \{ U \subseteq X | \forall x \in U \exists r > 0 \text{ s.t. } B_r(x) \subseteq U \} \]

is a topology, \( T_d \) on \( X \) induced by the metric \( d \).

Def 2.1 A topological space is a pair \((X,T)\) where \( T \)

is a topology on \( X \).

Def 2.2 Let \((X,T_X),(Y,T_Y)\) be two topological spaces.

A map \( f : X \rightarrow Y \) is continuous \( \text{w.r.t. } (T_X,T_Y) \) if

for any \( U \in T_Y \), \( f^{-1}(U) \in T_X \).

("The preimage of an open set is open")

You have checked that \( T = \{ X, \emptyset \} \) and \( T = \mathcal{P}(X) \) (\( \forall \) subsets of \( X \) are topologies on a set \( X \)). \( \mathcal{P}(X) \) is called the discrete topology (since \( \forall x \in X \), \( \{x\} \in \mathcal{P}(X) \)).

Ex X set

a cofinite topology on \( X \) in

\[ T_{cofin} = \{ U \subseteq X | X \setminus U \text{ is finite} \} \]

Exercise 1. If \( X \) is finite, \( T_{cofin} = \mathcal{P}(X) \)

2. If \( X \) is infinite \( \exists \) metric \( d : X \times X \rightarrow [0,\infty) \) so that

\[ T_d = T_{cofin}. \]
Def 2.3. Let \((X, \mathcal{T})\) be a topological space. A subset \(C \subseteq X\) is closed if \(X \setminus C\) is open (i.e., \(\mathcal{T}\)).

Exercise 2.1. Closed sets: \(C_1, C_2, C_{\mathcal{I}}\) closed \(\Rightarrow C_1 \cup C_2 \text{ is closed}\).
\(C \subseteq \mathcal{T}\) closed \(\Rightarrow \bigcap C \subseteq \mathcal{T}\) is closed.

Subspace topology

Let \((X, \mathcal{T})\) be a topological space, \(Y \subseteq X\) subset.

Problem: What's the smallest topology \(\mathcal{T}_Y\) on \(Y\) so that \(i: Y \hookrightarrow X, i(y) = y\) is continuous?

Solution: If \(i: (Y, \mathcal{T}_Y) \hookrightarrow (X, \mathcal{T}_X)\) is continuous then:
\[\forall U \in \mathcal{T}_X \text{ we must have } \mathcal{T}_Y \ni i^{-1}(U) = U \cap Y\]

So set
\[\mathcal{T}^\text{sub}_Y = \{ V \subseteq Y \mid \exists U \in \mathcal{T}_X \text{ st. } Y \cap U = V\}\]

Claim: \(\mathcal{T}^\text{sub}_Y\) is a topology.

Check:
1) \(\emptyset = X \cap \emptyset \subseteq Y\), \(\emptyset = \emptyset \cap Y \subseteq \mathcal{T}^\text{sub}_Y\).
2) If \(V_1, V_2 \subseteq \mathcal{T}^\text{sub}_Y\), \(\exists U_1, U_2 \subseteq X\) with \(V_i = Y \cap U_i\)
   \[V_1 \cap V_2 = (Y \cap U_1) \cap (Y \cap U_2) = Y \cap (U_1 \cap U_2)\]
   \(U_1 \cap U_2 \subseteq \mathcal{T}_X\) since \(\mathcal{T}_X\) is a topology \(\Rightarrow V_1 \cap V_2 \subseteq \mathcal{T}^\text{sub}_Y\).
3) If \(V \subseteq \mathcal{T}^\text{sub}_Y\) then \(\exists U \subseteq \mathcal{T}_X\) with \(V = Y \cap U\)
   \[\forall V \subseteq \mathcal{T}_Y \Rightarrow \forall U \subseteq \mathcal{T}_X \text{ st. } V = Y \cap U\]
   \[U \cap V \subseteq \mathcal{T}_X, \quad U \cap V \subseteq \mathcal{T}^\text{sub}_Y\]

\[\forall \text{ top space } (X, \mathcal{T}_X) \text{ and map } f: (Y, P(Y)) \to (X, \mathcal{T}_X)\]

is continuous: \(\forall U \subseteq \mathcal{T}_X, \quad f^{-1}(U) \subseteq P(Y)\)

Def 2.4 We call \(\mathcal{T}^\text{sub}_Y\) the \textit{subspace topology} on \(Y\).

Remark: Suppose \((X, \mathcal{T}_X)\) top space, \(Y \subseteq X\), \(\mathcal{T}_Y\) topology

so that \(i: (Y, \mathcal{T}_Y) \hookrightarrow (X, \mathcal{T}_X)\) is continuous.
Then \( \forall U \in J_x, i^*(U) = U \cap Y \in J_y \)

\[ \Rightarrow J^\text{sub}_y \subseteq J_y \]

\( \therefore J^\text{sub}_y \) is the smallest topology so that \( i : (Y, J_y) \to (X, J_x) \) is continuous.

**Proposition 2.5** Let \( \{J_x\}_{x \in A} \) be a family of topologies on a set \( X \).

Then \( J = \bigcap_{x \in A} J_x \) is a topology on \( X \).

**Proof**

\( \phi \in J_x \forall x \Rightarrow \phi \in \bigcap_{x \in A} J_x \).

Similarly, \( X \in \bigcup_{x \in A} J_x \).

If \( U, V \in J \), then \( U \cap V \in J_x \forall x \Rightarrow U \cap V \in \bigcap_{x \in A} J_x = J \).

If \( U \cup B \subseteq J \) then \( \cup_{B \in B} S \subseteq J_x \forall x \Rightarrow U \cup B \cap J_x \forall x \Rightarrow (U \cup B) \subseteq J = J \).

**Consequences**

Let \( X \) be a set, \( A \subseteq \mathcal{P}(X) \), i.e., a collection of subsets of \( X \).

Let \( A = \{ J \text{ topology on } X \mid A \subseteq J \} \)

collection of topologies \( J \) on \( X \) so that all elements of \( A \) are open with respect to \( J \).

Then \( J^\# = \bigcap A \) is a topology on \( X \) by 2.5.

And it's the smallest topology so that all elements of \( A \) are open; the topology generated by \( A \).

Since the construction of \( J^\# \) is fairly abstract and general, it is hard to see what \( J^\# \) may actually be.

**Two special cases:**
Def 2.6 A collection \( \mathcal{B} \subset \mathcal{P}(X) \) is called a basis for a topology on \( X \) if

1. \( \bigcup \mathcal{B} = X \)
2. \( \forall B_1, B_2 \in \mathcal{B}, \ B_1 \cap B_2 \) is a union of elements of \( \mathcal{B} \)

\( \mathcal{T}_{\mathcal{B}} := \text{topology generated by } \mathcal{B} \).

Ex. \( (X, d) \) metric space \( \mathcal{B} = \{ B_r(x) \mid x \in X, r > 0 \} \).

\( \forall z \in B_{r_1}(x_1) \cap B_{r_2}(x_2) \exists \rho > 0 \text{ s.t. } B_{\rho}(z) \subset B_{r_1}(x_1) \cap B_{r_2}(x_2) \)

\( \Rightarrow B_{r_1}(x_1) \cap B_{r_2}(x_2) = \text{union of open balls} \).

\( \Rightarrow \mathcal{B} \) is a basis for a topology on \( X \).

Prop 2.7 \( \mathcal{T}_\mathcal{B} = \{ \bigcup X \mid U \text{ in a union of elements of } \mathcal{B} \} \).

Proof: Clearly \( \mathcal{B} \subset \mathcal{T}_\mathcal{B} \).

Also, if \( \mathcal{T} \) is any topology and \( \mathcal{B} \subset \mathcal{T} \) then \( \mathcal{T}_\mathcal{B} \subset \mathcal{T} \).

\( \mathcal{T}_\mathcal{B} \) is the smallest topology containing \( \mathcal{B} \).

(provided we know it is a topology)

Is it?

\( \varnothing \) in the union of the empty collection of elements of \( \mathcal{B} \)

\( X = \bigcup \mathcal{B} \)

if \( U = \bigcup_{b \in \mathcal{B}} B_a \), \( V = \bigcup_{b \in \mathcal{B}} B_b \) Then \( U \cup V = \bigcup_{b \in \mathcal{B}} \text{elements of } \mathcal{B} \)

\( \Rightarrow U \cup V \in \mathcal{T}_\mathcal{B} \)

if \( U_a = \bigcup_{b \in A} B_{ar} \) Then \( \bigcup_{a \in A} U_a = \bigcup U_a \in \mathcal{T}_\mathcal{B} \).

\( \Rightarrow \mathcal{T}_\mathcal{B} \) is a topology.
Def 2.3 Let $X$ be a set. $\mathcal{S} \subseteq \mathcal{P}(X)$ is a subbasis for a topology on $X$ if $\bigcup \mathcal{S} = X$.

Prop 2.9 Then $\mathcal{T}_\mathcal{S} = \{ U \subseteq X \mid U$ is the union of finite intersections of elements of $\mathcal{S} \}$

Proof Similar to proof of 2.7.