Recall \( X \) is compact \( \iff \forall \) collection of closed subsets \( \{ C_\alpha \}_{\alpha \in A} \) with \( C_\alpha \cap C_\beta = \emptyset \text{ for } \alpha \neq \beta \), we have \( \bigcap_{\alpha \in A} C_\alpha \neq \emptyset \).

- \( p \in X \) is a cluster point of a net \( (x_\lambda)_{\lambda \in \Lambda} \iff \forall \) nbd \( W \) of \( p \) and \( \lambda_0 \in \Lambda \) \( \exists \lambda \in \Lambda \) with \( \lambda_0 < \lambda \) and \( x_\lambda \in W \).

Prop 11.1 p \( \in \) a cluster point of \( (x_\lambda)_{\lambda \in \Lambda} \) \( \iff \exists \) a subnet \( (x_{\lambda_m})_{m \in M} \)

converging to \( p \).

Proof (\( \Rightarrow \)). Suppose \( p \) is a cluster point of \( (x_\lambda)_{\lambda \in \Lambda} \). Let \( M = \{ (\lambda, W) \mid x_\lambda \in W, \text{ nbd of } p, x_\lambda \in W \} \).

\( (\lambda, W) < (\lambda', W') \iff \lambda < \lambda' \) and \( W \supset W' \).

Define \( \psi : M \to \Lambda \) by \( \psi(\lambda, W) = \lambda \).

Clearly \( (\lambda, W) < (\lambda', W') \implies \psi(\lambda, W) < \psi(\lambda', W') \).

Since \( p \) is a cluster point of \( (x_\lambda)_{\lambda \in \Lambda} \), \( \forall \lambda \in \Lambda \) and \( \forall \) nbd \( W \) of \( p \), \( \exists \lambda \in \Lambda \) with \( \lambda_0 < \lambda \) and \( x_\lambda \in W \).

Then \( (\lambda, W) \in M \) and \( \lambda_0 < \lambda \iff \psi(\lambda, W) = \lambda \).

\( \implies (x_{\lambda_m})_{m \in M} \) is a subnet of \( (x_\lambda)_{\lambda \in \Lambda} \) by def of a subnet.

We now check that \( x_{\lambda_m} \to p \).

Since \( p \) is a cluster point \( \forall \) nbd \( W \) of \( p \), \( \exists \lambda_1 \in \Lambda \) with \( x_{\lambda_1} \in W \).

For any \( (\lambda_1, W) \in M \) with \( (\lambda_1, W) < (\lambda, W) \) we have \( \lambda_1 < \lambda \) and \( x_{\lambda_1} \in W \).

Hence \( \exists m \in M \) with \( (\lambda_1, W) \leq _M \lambda \), \( x_{\lambda_m} \in W \).

\( \psi(\lambda_1, W) = \lambda_1 \).

Fix a nbd \( W \) of \( p \) and \( \lambda_0 \in \Lambda \). Want to find \( \lambda \) with \( \lambda_0 < \lambda \) and \( x_\lambda \in W \).

Since \( x_{\lambda_m} \to p \), \( \exists m \in M \) s.t. \( m \leq M \Rightarrow x_{\lambda_m} \in W \).
Since \((x_{\lambda n})_{\lambda n} \subseteq M \) is a subnet of \((x_{\lambda})_{\lambda \in \Lambda}\), \(\exists \mu \) s.t. \(\lambda_0 < \lambda_1 \leq \mu\).

Now \(\exists \mu_2 \in M\) with \(\mu_0 < \mu_2\) and \(\mu_1 < \mu_2\).
Then for \(\forall \mu\) with \(\mu_1 < \mu\) we have
\[x_{\mu_1} \in W\text{ and } x_{\mu_2} < x_{\mu_1} < x_{\mu_2} < x_{\lambda_0}.
\]

\[
\text{Prop 11.2} \quad X \text{ is compact } \iff \text{ every net } (x_{\lambda})_{\lambda \in \Lambda}\text{ in } X \text{ has a cluster point hence a converging subnet.}
\]

First, a definition: let \((x_{\lambda})_{\lambda \in \Lambda}\) be a net, \(\lambda_0 + \Lambda\). The \(\lambda_0\)-tail of \((x_{\lambda})_{\lambda \in \Lambda}\) is
\[T_{\lambda_0} := \{x_{\lambda} \mid \lambda > \lambda_0\}.
\]

Next observe that \(\forall \lambda_1 \in \Lambda\) the set \(\{T_{\lambda_i} \}_{\lambda_i \in \Lambda}\) has
F. I. P. : \(\forall \lambda_1, \ldots, \lambda_k \in \Lambda\) \(\exists \mu \in \Lambda\) with
\[\lambda_i < \mu \quad (i = 1, \ldots, k)
\]
\[\implies T_{\lambda_1} \cap \ldots \cap T_{\lambda_k} \neq \emptyset
\]

Proof of 11.2 (\(\Rightarrow\)) Suppose \((x_{\lambda})_{\lambda \in \Lambda}\) is a net in a compact space \(X\). Consider the collection \(\{T_{\lambda} \}_{\lambda \in \Lambda}\) of closures of tails of \((x_{\lambda})_{\lambda \in \Lambda}\). It has F. I. P. \(\Rightarrow \bigcap_{\lambda \in \Lambda} T_{\lambda} \neq \emptyset\)

The \(\forall \mu \in \Lambda\) \(\forall B \in \mathcal{B}\) \(\forall x \in \mathcal{B}\) \(\exists \mu \) with \(\lambda \geq \lambda \) and \(x \in B\).

The \(\forall \mu \in \Lambda\) \(\forall x \in \mathcal{B}\) \(\forall \mu \in \Lambda\) \(\exists \mu \) with \(\lambda \geq \lambda \) and \(x \in B\).
Suppose every net in $X$ has a cluster point.

Let $G$ be a collection of closed subsets of $X$ with F. I. P.

Want to show: $\bigcap_{C \in G} C \neq \emptyset$.

Let $G = \{ C_n \mid n \in \mathbb{N}, \, C_1, \ldots, C_n \in G \}$.

- set of finite intersections of elements of $G$.

Since $G$ has F. I. P., so does $G$.

Direct $G$ by reverse inclusion: $G_1 \subseteq G_2 \Rightarrow G_1 \cup G_2 \subseteq G_2$.

It's a preorder and $\forall G_1, G_2$

$$G_1, G_2 \subseteq G_1 \cap G_2$$

$\Rightarrow \ (G, \subseteq)$ is a directed set.

Now for each $G \in G$ choose $x_G \in G$. By assumption

$(x_G)_{G \in G}$ has a cluster point $p$, i.e. $\forall \text{ nbd } W$ of $p$ $\forall G \subseteq G'$ with $G \subseteq G'$ and $x_G \in W$.

$(\Rightarrow G \subseteq G')$

Then $G \cap W \supseteq G' \cap W \ni x_{G'} \Rightarrow G \cap W \neq \emptyset \forall G \in G$.

$\Rightarrow \forall G \in G'$, $p \in G = G$

$\Rightarrow p \in \bigcap_{G \in G'} G \subseteq \bigcap_{C \in G} C$

$\therefore \bigcap_{C \in G} C \neq \emptyset$.

To prove arbitrary products of compact spaces are compact we'll use Zorn's lemma:

If $X$ is a (nonempty) partially ordered set (poset) and if every chain in $X$ has an upper bound then $X$ has a maximal element.
Here are the definitions of various words:

Recall that a relation \( \leq \) on a set \( X \) is a preorder if

1. \( x \leq x \) for all \( x \in X \) (\( \leq \) is reflexive)

2. \( x \leq y \) and \( y \leq z \) imply \( x \leq z \) (\( \leq \) is transitive)

It is a partial order if it is also anti-symmetric:

\[ x \leq y \text{ and } y \leq x \Rightarrow x = y. \]

A pair \((X, \leq)\), where \( \leq \) is a partial order, is called a poset.

A subset \( Y \) of a poset \((X, \leq)\) is a chain if it is totally ordered:
\[ \forall y_1, y_2 \in Y \text{ either } y_1 \leq y_2 \text{ or } y_2 \leq y_1. \]

Example: The set of subsets \( P(S) \) of a set \( S \) is a poset under reverse inclusion. A chain \( Y \) in \( P(S) \) is a collection of nested subsets:
\[ \forall A, B \in Y \text{ either } A \subseteq B \text{ or } B \subseteq A. \]

An element \( x \) of a poset \((X, \leq)\) is maximal if
\[ x \leq y \Rightarrow y = x. \]

An upper bound of a subset \( Y \) of a poset \((X, \leq)\)
is \( u \in X \) with \( y \leq u \) for all \( y \in Y \).

Example: \( S \) is a set, \( F \subseteq P(S) \) is a collection of subsets of \( X \) with F.I.P., let \( X = \{ F \subseteq P(S) \mid F \text{ has F.I.P. and } \emptyset \subseteq F \} \). Then \( X \) is a poset under inclusion.

We'll show: \( X \) has a maximal element.