Recall: The definition of open sets in $\mathbb{R}^n$ is done in several steps:

1. First one defines a norm $\| \cdot \|$ on $\mathbb{R}^n$.
2. Norms determine distance $d$.
3. $d$ allows us to define open balls.
4. Open balls give us open sets.

Here are the details:

1. Given $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, the Euclidean norm $\| \cdot \|$ is defined by
   \[ \| x \| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}. \]

   **Properties**
   
   a) $\| x \| \geq 0$ and $\| x \| = 0 \iff x = 0$.
   
   b) $\| \lambda x \| = |\lambda| \| x \|$, $\forall x \in \mathbb{R}^n$, $\forall \lambda \in \mathbb{R}$.
   
   c) $\| x + y \| \leq \| x \| + \| y \|$, (triangle inequality).

2. We define $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ by
   \[ d(x, y) = \| x - y \|. \]

   **Note:** $d(x, y) = d(y, x) + xy$.

   Properties (a)–(c) of the norm $\| \cdot \|$ easily translate into properties of $d$:

   (4a) $\Rightarrow$ (2a): $d(u, v) \geq 0$ and $d(u, v) = 0 \iff u = v$.
   
   (4c) $\Rightarrow$ (2c): $d(u, v) \leq d(u, w) + d(w, v)$, $\forall u, v, w \in \mathbb{R}^n$. 

   ![Diagram](image-url)
Definition 1.1: A set $X$ together with a function 
\[ d : X \times X \to [0, \infty) \]
so that
1. $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$ and
3. $d(x, y) \leq d(x, z) + d(y, z) \quad \forall x, y, z \in X$

is called a **metric space**.

Thus $\mathbb{R}^n$ with $d(x, y) := \| x - y \| = \left( \sum (x_i - y_i)^2 \right)^{\frac{1}{2}}$

is a metric space.

Now, given a metric space $(X, d)$ we define an open ball $B_r(x)$ centered at $x \in X$ of radius $r > 0$ by

\[ B_r(x) := \{ y \in X \mid d(x, y) < r \} \]

**Example**

1. $X = \mathbb{R}$, $d(x, y) = |x - y|$ 
   
   \[ B_r(x) = (x - r, x + r) \]

2. $X = \mathbb{R}^2$, $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

   \[ B_1(0) = \]

   ![Diagram](attachment:image.png)

Observation: Fix a metric space $(X, d)$, $x \in X$, $r > 0$.

For any $y \in B_r(x)$ $\exists \rho > 0$ so that $B_\rho(y) \subseteq B_r(x)$.

**Reason:**

Take $\rho = r - d(x, y)$.

Then $z \in B_\rho(y) \iff d(y, z) < \rho = r - d(x, y) \iff d(x, z) < r$.

$\Rightarrow d(x, z) < d(x, y) + d(y, z) < r \Rightarrow z \in B_r(x)$. $\square$
Definition 1.2 Let \((X, d)\) be a metric space.

A subspace \(U \subseteq X\) is open if \(\forall x \in U \exists r > 0\) so that \(B_r(x) \subseteq U\).

Examples
- \(X\) is open;
- \(\emptyset \subseteq X\) is open;
- \(\forall x \in X \exists r > 0\) \(B_r(x) \subseteq X\) is open.

Exercise Let \((X, d)\) be a metric space. Then

(i) If \(U, V \subseteq X\) are open (in the sense of 1.2) so is \(U \cup V\):

\[\forall x \in U \cup V \exists r > 0\text{ so that } B_r(x) \subseteq U \cup V.\]

(ii) If \(\\{U_a\}_{a \in A}\) is a collection of open sets, then so is \(\bigcup U_a\).

Continuity

Recall the \(\varepsilon-\delta\) definition of continuity:

\[f : \mathbb{R} \to \mathbb{R}\] is continuous if \(\forall x_0 \in \mathbb{R}\)

\[\forall \varepsilon > 0 \exists \delta = \delta(f, x_0)\text{ so that }\]

\[|x_0 - x| < \delta \implies |f(x_0) - f(x)| < \varepsilon.\]

This is easy to generalize to any map between metric spaces:

Definition Let \((X, d_X)\), \((Y, d_Y)\) be two metric spaces.

\[f : X \to Y\] is continuous if \(\forall x_0 \in X \exists \delta = \delta(f, x_0)\) so that

\[d_Y(f(x_0), f(x)) < \varepsilon \implies d_X(x_0, x) < \delta.\]

Two points that are supposed to motivate the notion of a topology:
1. Two different metrics may define the same collection of open sets.

2. The notion of continuity doesn't really depend on metrics; it only depend on open sets.

Example. \( X = \mathbb{R}^2 \), \( \| x \|_\infty = \max \{ |x_1|, |x_2| \} \) is a norm.

\[
d_\infty : \mathbb{R}^2 \times \mathbb{R}^2 \to (0, \infty), \quad d_\infty (x, y) = \| x - y \|_\infty
\]

is a metric.

Open balls are squares:

Open sets are same as the ones defined by the Euclidean distance.

Lemma 1.4. Let \((X, d_X), (Y, d_Y)\) be two metric spaces. Then \( f : X \to Y \) is continuous (in the sense of Def 1.3) \( \iff \forall U \subseteq Y \text{ open, } f^{-1}(U) \subseteq \text{ open in } X \).

Proof. (\( \Rightarrow \)) Suppose \( U \subseteq Y \text{ open} \), \( y_0 \in f^{-1}(U) \).

Want to show: \( \exists \delta > 0 \text{ s.t. } B_{\delta}(x_0) \subseteq f^{-1}(U) \).

Let \( y_0 = f(x_0) \). Since \( U \subseteq Y \text{ open} \), \( \exists \epsilon > 0 \text{ s.t. } B_{\epsilon}(y_0) \subseteq U \).

Since \( f \) is continuous, \( \exists \delta > 0 \text{ s.t. } d_Y(x, x_0) < \delta \Rightarrow d_X(f(x), f(x_0)) < \epsilon \).

Thus, \( x \in B_{\delta}(x_0) \Rightarrow d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \epsilon \).

\[
\Rightarrow f(B_{\delta}(x_0)) \subseteq U
\]

\[
\Rightarrow B_{\delta}(x_0) \subseteq f^{-1}(U) \]

\( \Rightarrow f^{-1}(U) \text{ is open since } x_0 \text{ is arbitrary.} \)
Suppose \( \forall U \subseteq Y \) open, \( f^{-1}(U) \) is open in \( X \).

Let \( x_0 \in X \) be a point and \( \varepsilon > 0 \).

We have seen: \( U = B_\varepsilon(f(x_0)) \) is open in \( Y \).

By assumption \( f^{-1}(B_\varepsilon(f(x_0))) \) is open in \( X \).

\[ \exists \delta > 0 \text{ s.t. } B_\delta(x_0) \subseteq f^{-1}(B_\varepsilon(f(x_0))) \]

\[ \forall x \in X \text{ with } d_X(x, x_0) < \delta \text{ we have } x \in B_\delta(x_0) \]

\[ \Rightarrow x \in f^{-1}(B_\varepsilon(f(x_0))) \]

\[ \Rightarrow f(x) \in B_\varepsilon(f(x_0)) \]

\[ \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon. \quad \Box \]

**Def 1.5 (Topology)**

A topology \( \mathcal{T} \) on a set \( X \) is a collection of subsets of \( X \) called "open sets" (i.e., \( \mathcal{T} \subseteq \mathcal{P}(X) \)), \( \mathcal{P}(X) \) = set of all subsets of \( X \), s.t.

1. \( X, \emptyset \in \mathcal{T} \)
2. \( U, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T} \)
3. \( \{ U_a \subseteq \mathcal{A} \in \mathcal{T} \Rightarrow \bigcup U_a \in \mathcal{T} \).