MATH 535  HW #4  Due, March 7, 2011.

#1  Let $X$ be a topological space. Let $\{F_{\alpha}\}$ be a family of closed compact subspaces of $X$. Show that $\bigcap_{\alpha} F_{\alpha}$ is compact. Give an example to show that closedness is necessary. Hint: For what kind of spaces may compact subspaces be non-closed?

#2  Prove that products of Hausdorff spaces are Hausdorff.

#3  Suppose $f, g : X \to Y$ are two continuous maps. (i) Show that if $Y$ is Hausdorff then the set $\{f=g\} = \{x \in X \mid f(x) = g(x)\}$ is closed. (ii) Show that the assumption that $Y$ is Hausdorff is necessary. Hint: Consider $(f, g) : X \times X \to X \times Y$.(iii) Show that $f, g : X \to Y$ are as above, $Y$ is Hausdorff and $f = g$ on a dense subset of $X$, then $f = g$.

#4  Given a metric $d : X \times X \to [0, \infty)$ define $d' : X \times X \to [0, 1]$ by

$$d'(x, y) = \begin{cases} 1 & \text{if } d(x, y) \geq 1 \\ \frac{1}{d(x, y)} & \text{if } d(x, y) < 1 \end{cases}$$

Prove that $d'$ is a metric and that it defines the same topology as $d$.

#5  Let $X$ be a compact metric space, $A \subseteq X$ closed, $V \subseteq X$ open with $A \subseteq V$. Show $\exists \varepsilon > 0$ so that $B_{\varepsilon}(a) \subseteq V, \forall a \in A$. Hint: Lebesgue lemma.

#6  Show that a totally bounded metric space is separable.

#7  Let $(X, d_X), (Y, d_Y)$ be two metric spaces, $f : X \to Y$ continuous. Show that if $X$ is compact then $\forall \varepsilon > 0 \exists \delta > 0$ s.t. for all $x, x' \in X$

$$d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon.$$