Lemma: Let $\{U, V\}$ be an open cover of a manifold $M$, $i_U : U \cap V \rightarrow U$, $i_V : U \cap V \rightarrow V$, $i_{U \cap V} : U \cap V \rightarrow M$ the corresponding maps. Then, take $N$, the sequence

$$0 \rightarrow \Omega^k(M) \stackrel{i^*}{\rightarrow} \Omega^k(U) \oplus \Omega^k(V) \stackrel{i^* + i^*}{\rightarrow} \Omega^k(U \cap V) \rightarrow 0$$

is exact.

Proof: 1. If $\omega \in \Omega^k(M)$, $\omega|_U = 0$, $\omega|_V = 0$ then $\omega = 0$. ⇒ The sequence is exact at $\Omega^k(M)$.

2. If $\sigma \in \Omega^k(U)$, $\tau \in \Omega^k(V)$, $\iota^* \sigma = 0|_{U \cap V} = 0$ and $\iota^* \tau = 0|_{U \cap V} = 0$ then $\exists \omega \in \Omega^k(M)$ so that $\omega|_U = \sigma$ and $\omega|_V = \tau$. ⇒ The sequence is exact at $\Omega^k(U) \oplus \Omega^k(V) \rightarrow \Omega^k(U \cap V)$.

3. (exactness at $\Omega^k(U \cap V)$). Suppose $\beta \in \Omega^k(U \cap V)$. We want to find $\sigma \in \Omega^k(U)$ and $\tau \in \Omega^k(V)$ so that $\sigma|_{U \cap V} - \tau|_{U \cap V} = \beta$. Choose a partition of 1 $\{p_U, p_V\}$ subordinate to $\{U, V\}$. $p_U \beta$ is supported in $V$ so extends by 0 to a $k$-form on $U$, which we again denote by $p_U \beta$. Similarly $p_V \beta$ extends to a $k$-form on $V$. Moreover,

$$\beta = (p_U \beta)|_{U \cap V} + (p_V \beta)|_{U \cap V} = i^*_V (p_U \beta) + i^*_U (p_V \beta) = i^*_U (p_V \beta) - i^*_V (-p_U \beta).$$

Recall: A sequence $0 \rightarrow A^0 \rightarrow A^1 \rightarrow A^2 \rightarrow \cdots$ of vector spaces and linear maps is a cochain complex if $d: A^k \rightarrow A^{k+1}$ is $d \circ d = 0$.

By definition its $k^{th}$ cohomology is

$$H^k(A) = \ker d : A^k \rightarrow A^{k+1} / \im d : A^{k-1} \rightarrow A^k.$$

And the (total) cohomology is

$$H^* = \bigoplus_k H^k(A).$$
Let \( A^n \rightarrow \cdots \rightarrow A^0 \rightarrow 0 \) and \( B^n \rightarrow \cdots \rightarrow B^0 \rightarrow 0 \) be two chain complexes. A map of chain complexes \( f : A^* \rightarrow B^* \) is a collection of maps \( f^k : A^k \rightarrow B^k \) \((\forall k)\) so that

\[
\begin{align*}
A^{k+1} & \xrightarrow{f} B^{k+1} \\
\downarrow d & \quad \downarrow d \\
A^k & \xrightarrow{f} B^k
\end{align*}
\]

commutes: \( f \circ d = d \circ f \).

A sequence of chain complexes \( 0 \rightarrow A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \rightarrow 0 \) is a short exact sequence if for each \( k \) the sequence

\[
0 \rightarrow A^k \xrightarrow{f} B^k \xrightarrow{g} C^k \rightarrow 0
\]

is short exact.

**Lemma 2** A short exact sequence \( 0 \rightarrow A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \rightarrow 0 \) of chain complexes induces an exact sequence

\[
0 \rightarrow H^0(A) \xrightarrow{f^*} H^0(B) \xrightarrow{g^*} H^0(C) \xrightarrow{\delta} H^1(A) \xrightarrow{f^*} \cdots
\]

The long exact sequence in cohomology.

(Here \( f : H^k(A) \rightarrow H^k(B) \) is defined by \( f([a]) = [f(a)] \)) \( \forall [a] \in H^k(A) \),

where on the right the \( f \) is the map \( f : A^k \rightarrow B^k \); so I am abusing notation by using the symbol “\( f \)” for two rather different things.)

**Proof**

\[
\begin{array}{cccccc}
0 & \rightarrow & A^{k+2} & \xrightarrow{f} & A^{k+1} & \xrightarrow{d} \\
\downarrow f & \quad & \downarrow f & \quad & \downarrow d & \\
0 & \rightarrow & A^k & \xrightarrow{f} & B^k & \xrightarrow{g} \\
\downarrow f & \quad & \downarrow g & \quad & \downarrow d & \\
0 & \rightarrow & A^{k-1} & \xrightarrow{f} & B^{k-1} & \xrightarrow{g} \\
\end{array}
\]

\[
\begin{array}{cccccc}
\delta & \rightarrow & C^{k+1} & \xrightarrow{0} \rightarrow \\
\downarrow d & \quad & \downarrow d & \quad & \downarrow d & \\
0 & \rightarrow & C^k & \xrightarrow{0} \rightarrow \\
\downarrow d & \quad & \downarrow d & \quad & \downarrow d & \\
0 & \rightarrow & C^{k-1} & \xrightarrow{0} \rightarrow \\
\end{array}
\]
The first step is to define \( \delta : H^k(C) \rightarrow H^{k+1}(A) \). Pick \( \omega \in C^k \) with \( d\omega = 0 \). Since \( g : B^k \rightarrow C^k \) is onto, \( \exists \gamma \in B^k \) with \( g(\gamma) = \omega \). Then, \( g(d\gamma) = d g(\gamma) = d\omega = 0 \). Since \( \ker g = \text{im} f \), \( \exists \xi \in A^{k+1} \) with \( f(\xi) = d\gamma \). Moreover, \( f(d\xi) = d f(\xi) = d( d\gamma) = 0 \). Since \( f \circ \omega = 1 \), \( d\xi = 0 \). We define

\[
\delta(\omega) = [\xi].
\]

We need to check that \( \delta \) doesn't depend on the choices of \( \omega \) and \( \gamma \) (since \( f \circ \omega = 1 \), there was no choice with \( \xi \) ). Suppose \( [\omega] = [\omega'] \), i.e. \( \omega - \omega' = d\xi \) for some \( \xi \in C^{k-1} \). As in the case of \( \omega \), we choose \( \gamma' \) with \( g(\gamma') = \omega' \) and \( \xi' \) with \( f(\xi') = d\gamma' \). Then \( g(\gamma - \gamma') = g(\gamma) - g(\gamma') = \omega - \omega' = d\xi \). Since \( g \) is onto, \( \xi = d\theta \) for some \( \theta \in B^k \). Since \( g(d\theta) = d g(\theta) \), \( g(\gamma - \gamma' - d\theta) = 0 \). Hence, by exactness at \( B^k \), \( \gamma - \gamma' - d\tau = f(\psi) \) for some \( \psi \in A^k \). \( \Rightarrow f(\xi - \xi') = d\eta - d\eta' = d(\eta - \eta') = d(\gamma - \gamma' - d\theta) = d(\gamma - \gamma' - d\tau) = d f(\psi) = f(d\psi) \). Since \( f \circ \omega = 1 \), \( \xi - \xi' = d\psi \). \( \Rightarrow [\xi] = [\xi'] \). Therefore \( \delta : H^k(C) \rightarrow H^{k+1}(A) \) is well-defined.

Next, we check exactness of \( H^k(B) \xrightarrow{\delta} H^k(C) \xrightarrow{\delta} H^{k+1}(A) \xrightarrow{\delta} H^k(C) \).

Suppose \( \delta(\omega) = 0 \), i.e. \( \delta(\omega) = [d\xi] \) for some \( \xi \in A^k \). We want to show that \( [\omega] = g([\gamma]) \) for some \( [\gamma] \in H^k(B) \).

That is, \( \exists \gamma \in B^k \) with \( d\gamma = 0 \) and \( [g(\gamma)] = [\omega] \).

By definition of \( \delta \), \( \exists \gamma' \in B^k \) such that \( g(\gamma') = \omega \) and \( d\gamma' = f(d\xi) \). Then \( d\gamma' = f(d\xi) = d f(\xi) \Rightarrow d(\gamma' - f(\xi)) = 0 \). Let \( \gamma = \gamma' - f(\xi) \).

Then \( g(\gamma) = g(\gamma') - g(f(\xi)) = g(\gamma') - 0 = \omega \) and \( d\gamma = 0 \) by construction. \( \Rightarrow [\gamma] \) makes sense and \( g([\gamma]) = [\omega] \).
This proves: \( \ker \delta \leq \text{img.} \)

Conversely, consider \( g(y) \) for some \( y \in B^k \) with \( dy = 0 \). We want to show, \( g(y) = 0 \). But \( g(y) = Lg(y) \). And, by definition of \( \delta \),

\[ \delta(Lg(y)) = 0 \]

is the class of the unique \( \xi \in \mathbb{A}^{k-1} \) with \( f(\xi) = dy \). Since \( dy = 0 \) and \( f \) is 1-1, \( \xi = 0 \). Thus \( Lg(y) = 0 \). \( \Rightarrow \)

\[ \text{img.} \leq \ker \delta. \]

Therefore \( \text{img.} = \ker \delta \).

We check exactness at \( H^k(A) \Rightarrow H^k(C) \Rightarrow H^{k+1}(A) \Rightarrow H^{k+1}(B) \).

For \( [\omega] \in H^k(C) \), \( f(\delta(\omega)) = f(\delta) = Lf(\xi) = Lf(\delta) \), where \( f(\xi) = dy \) and \( g(y) = \omega. \Rightarrow f(\delta(\omega)) = Lf(\delta) = L(\delta y) = 0. \Rightarrow \delta y = 0 \) or \( \ker f \supset \text{img.} \delta \).

Conversely, suppose \( \xi \in \ker f \). Then \( f(\xi) = dy \) for some \( y \in B^k \). Let \( \omega = g(y) \). Then, \( d\omega = dg(y) = g(dy) = g(f(\xi)) = 0 \), so \( [\omega] \) makes sense. And, by definition of \( \delta \), \( \delta(\omega) = [\xi] \).

Hence \( \ker f \leq \text{img.} \delta \).

Therefore \( \ker f = \text{img.} \delta \).

Finally we check exactness of \( H^k(A) \Rightarrow H^k(B) \Rightarrow H^k(C) \Rightarrow H^{k+1}(B) \).

For \( [\xi] \in H^k(A) \), \( g(f(\xi)) = g(f(\xi)) = 0 \) since \( g \circ f = 0 \) as a map from \( A^k \) to \( C^k \). \( \Rightarrow g \circ f = 0 \) as a map from \( H^k(A) \) to \( H^k(C) \).

Remains to show: if \( g(y) = 0 \) then \( [y] = f(\xi) \) for some \( \xi \in H^k(A) \).

Now, \( g(y) = 0 \Rightarrow g(y) = d\xi \) for some \( \xi \in C^{k-1} \). Since \( g: B^{k-1} \rightarrow C^{k-1} \) is onto, \( \exists \theta \in B^{k-1} \) with \( g(\theta) = \xi \). Then \( g(y + d\theta) = g(y) - g(d\theta) = 0 \).
\[ \delta - d\varphi(\vartheta) = d\varphi - d\varphi = 0. \]

By exactness of \( \beta^k \mapsto \beta^k \to \alpha_k \) at \( d^k \),
\( \eta - d\vartheta = f'(\xi) \) for some \( \xi \in \beta^k \).
Moreover, \( f'(d\xi) = df \circ f = d(f'(\xi)) = d(\eta - d\vartheta) = d\eta = 0. \)
Since \( f \) is 1-1, \( d\xi = 0. \) So \( \{\xi\} \) makes sense, and \( \{\xi\} \) makes sense, and \( f(\{\xi\}) = \{\eta - d\vartheta\} = \{\eta\}. \)

**Remark** What is \( \delta : H^k(\mathbb{R}^n) \to H^{k+1}(\mathbb{R}^n) \), concretely?

Say we have a class \( [\omega] \in H^k(\mathbb{R}^n) \) defined by some closed \( k \)-form \( \omega \in \Omega^k(\mathbb{R}^n) \). What form \( \xi \in \Omega^{k+1}(\mathbb{R}^n) \) represents \( \delta([\omega]) \)?

Recall that to define \( \delta([\omega]) \) we need to find \( \eta + \Omega^k(\mathbb{R}^n) = \Omega^k(\mathbb{R}^n) \otimes \Omega^1(\mathbb{R}^n) \) so that \( \tilde{\omega}, \tilde{\omega} = \omega \) and then find \( \xi \in \Omega^{k+1}(\mathbb{R}^n) \) with \( \xi|_u = d\eta|_u \), \( \xi|_v = d\eta|_v \). We have seen that \( \eta = (p_u \omega, -p_u \omega) \in \Omega^k(\mathbb{R}^n) \otimes \Omega^1(\mathbb{R}^n) \), where \( p_u, p_v \) is the partition of 1 subordinate to \( \{u, v\} \). So there is a well-defined closed \( k+1 \)-form \( \xi \in \Omega^{k+1}(\mathbb{R}^n) \) with

\[
\begin{align*}
\xi|_u &= d(p_u \omega) \\
\xi|_v &= -d(p_u \omega)
\end{align*}
\]