last time proved that if $\phi: P \rightarrow M$ is a principal $G$-bundle, then

$$\text{connection 1-forms} \rightarrow \text{horizontal invariant distributions}$$


Recall if $\pi: \Omega \rightarrow M$ is a submersion and $f: N \rightarrow M$ is any map of manifolds then $(f, \pi): N \times \Omega \rightarrow M \times M$ is transverse to the diagonal $\Delta_M = \{(x, y) \in M \times M \mid x = y\}$

$$\Rightarrow N \times \Omega \equiv \{(n, q) \in N \times \Omega \mid f(n) = \pi(q)\}$$

is an embedded submanifold of $N \times \Omega$.

It comes with two canonical maps $\text{pr}_i: N \times \Omega \rightarrow N$ and $\tilde{f} \equiv \text{pr}_2: N \times \Omega \rightarrow \Omega$ making the diagram

$$\begin{array}{ccc}
N \times \Omega & \xrightarrow{\tilde{f}} & \Omega \\
\downarrow \text{pr}_1 & & \downarrow \pi \\
N & \xrightarrow{f} & M
\end{array}$$

In fact $N \times \Omega \xrightarrow{\tilde{f}} \Omega$ has a universal property:

For any manifold $Z$ and a pair of maps $g_1: Z \rightarrow N$, $g_2: Z \rightarrow \Omega$, so that $g_2 \circ g_1 = f$ and $g_1, g_2$ commute, $\tilde{f} \equiv g_1: Z \rightarrow N \times \Omega$ and $\pi \circ \text{pr}_1 \circ g_1 \equiv g_2$.

So we have

$$\begin{array}{ccc}
Z & \xrightarrow{g_2} & \Omega \\
\downarrow g_1 & & \downarrow \pi \\
N & \xrightarrow{f} & M
\end{array}$$

commutes. $\tilde{f} \equiv g_1: Z \rightarrow N \times \Omega$.

So we have

$$\begin{array}{ccc}
Z & \xrightarrow{g_2} & \Omega \\
\downarrow g_1 & & \downarrow \pi \\
N & \xrightarrow{f} & M
\end{array}$$

commutes. $\tilde{f} \equiv g_1: Z \rightarrow N \times \Omega$.
Lemma 20.1 Suppose \( Q \rightarrow M \) is a fiber bundle with typical fiber \( F \). Then \( U \) is a manifold \( f : N \rightarrow M \),

\[ \pi : N \times M Q \rightarrow N \]

is also a fiber bundle with typical fiber \( F \).

**Proof** We need to check that \( \pi \) is locally trivial. So suppose \( U \subset M \) is an open set with \( Q \mid U \rightarrow U \) trivial and \( \varphi : Q \mid U \rightarrow U \times F \) a trivialization. Then \( f^{-1}(U) \subset N \) is open,

\[ f^{-1}(U) \times M Q \]

is an open submanifold of \( N \times M Q \) and the map \( \varphi : Q \mid U \rightarrow U \times F \) induces

\[ \hat{\varphi} : f^{-1}(U) \times M Q \rightarrow f^{-1}(U) \times M (U \times F) \]

Since \( \hat{\varphi} \) is a diffeo, \( \hat{\varphi} \) is a diffeo as the inverse \( \hat{\varphi}^{-1} : U \times F \rightarrow Q \mid U \)

induces the inverse of \( \hat{\varphi} \).

\[ \Rightarrow \hat{\varphi} \text{ is a local trivialization of } N \times M Q \rightarrow N \]

Corollary 20.2 If \( G \rightarrow P \rightarrow M \) is a principal \( G \)-bundle then \( \pi \) is a manifold \( f : N \rightarrow M \),

\( f : N \rightarrow M \) is a principal \( G \)-bundle.

**Proof** The right action of \( G \) on \( P \) induces a right action of \( G \) on \( f^* P = N \times M P \). Since a local trivialization

\[ P \mid U \rightarrow U \times C \]

is \( G \)-equivariant, the induced map

\[ f^{-1}(U) \times M (P \mid U) \rightarrow f^{-1}(U) \times M (U \times G) \]

is \( G \)-equivariant as well.

\[ \Rightarrow f^* P \rightarrow N \] is a principal \( G \)-bundle.

Corollary 20.3 Suppose \( N \rightarrow M \) a map of manifolds and \( G \rightarrow P \rightarrow M \) a principal \( G \)-bundle
For any connection 1-form $A \in \Omega'(p, \mathfrak{g})$,

$\tilde{f}^* A \in \Omega'(\tilde{f}^* p, \mathfrak{g})$ is a connection 1-form.

**Proof.** We need to check:

1. $(\tilde{f}^* A)(X \tilde{f} p) = X, \forall X \in \mathfrak{g}$
2. $\rho_a (\tilde{f}^* A) = \text{Ad}(a^{-1}) \tilde{f}^* A, \forall a \in G$

Now, since $\tilde{f}$ is $G$-equivariant, $x \in f^* p, \forall x \in \mathfrak{g}$

$\tilde{f}(x \cdot \exp t \mathfrak{g}) = \tilde{f}(x) \cdot \exp t \mathfrak{g}$

$\Rightarrow \quad d\tilde{f}_x (X_{\tilde{f}^* p} (x)) = X_{\tilde{f} \tilde{f}(x)}$

$\Rightarrow \quad (\tilde{f}^* A)_x (X_{\tilde{f}^* p} (x)) = A_{\tilde{f}(x)} (d\tilde{f}_x (X_{\tilde{f}^* p} (x)))$

$= A_{\tilde{f}(x)} X_{\tilde{f}} (\tilde{f}(x)) = X.$

**Note:** $\tilde{f}$ is $G$-equivariant means: $\forall x \in f^* p, a \in G$

$\tilde{f}(x \cdot a) = \tilde{f}(x) \cdot a$

or

$\tilde{f} \circ R_a = \tilde{f} \circ R_a$

action on $f^* p$ \hspace{1cm} action on $\tilde{f}$

$\Rightarrow \quad R_a \tilde{f}^* A = \tilde{f}^* (R_a^* A) = \tilde{f}^* (\text{Ad}(a^{-1}) A)

= \text{Ad}(a^{-1}) \tilde{f}^* A.$

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**Associated bundles.**

Let $G \to \mathbb{P} \to M$ be a principal G-bundle.

Suppose $F$ is a manifold with a left action of $G$,

$G \times F \to F, \quad (a, x) \mapsto a \cdot x.$

Then we have a left action of $G$ on $P \times F$ given by

$a \cdot (p, x) = (p \cdot a^{-1}, a \cdot x)$

Let $P \times^G F = (P \times F) / G$ denote the orbit space

$= \{ [p, x] \mid (p, x) \in P \times F \}.$

**Note:** Notation $P \times^G F$ is also used, but it clashes with fiber product.
notation]

The projection \( \mathcal{P} \times F \to \mathcal{P} \), \( (p,x) \to p \)
is \( G \)-equivariant, so it descends to a map of orbit spaces

\[
\mathcal{P} \times^G F \to \mathcal{P}/G \times M
\]

\[
(p,x) \mapsto (p) \mapsto \pi(p)
\]

**Lemma 20.4** Under the above assumptions, \( \mathcal{P} \times^G F \) is a manifold
and \( \pi: \mathcal{P} \times^G F \to M \), \( (p,x) \to \pi(p) \)
is a fiber bundle with typical fiber \( F \).

If \( F \) is a vector space and the action of \( G \) on \( F \) comes
from a representation \( \rho: G \to GL(F) \), then

\[
\mathcal{P} \times^G F \to M
\]
is a vector bundle.

**Aside** Given a left action of a group \( G \) on a manifold
\( Q \), what is a sufficient condition for \( Q/G \) to be
a manifold?

First of all we'd like to parameterize at least some of the orbits
by a manifold of correct dimension.

Secondly we'd like these local parameterizations to be
compatible.

So suppose we can find a collection \( \{ \Sigma_x \}_{x \in X} \) of
embedded submanifolds \( \Sigma_x \) that

\( \forall x \in \Sigma_x \quad G \cdot x \cap \Sigma_x = \{ x \} \)

\( G \cdot x \) orbit through \( x \).

\( \forall q \in Q \cap \Sigma_x \) so that \( G \cdot q \cap \Sigma_x \neq \emptyset \)

\( \forall x \in X \quad \Sigma_x \to Q \xrightarrow{\pi} Q/G \)

\( \pi \) onto

\( \forall x \in X \quad \Sigma_x \ni x \mapsto G \cdot x \cap \Sigma_x \)

The maps \( \pi^0(\Sigma_x) \cap \Sigma_x \to \pi^0(V_{\mathcal{P}}) \cap \Sigma \)
are \( \mathbb{C}^0 \).
Then the embeddings $\mathbb{C}^2 \to \mathbb{Q} \to \mathbb{Q}/G$ give $\mathbb{Q}/G$ the structure of a manifold.

$E$ $\mathbb{C}^2$ action on $E^2 \times_{\mathbb{Q}^\times} \mathbb{Q}$: $x \cdot (z_1, z_2) = (\lambda z_1, \lambda z_2)$

$E_1 = \{(z_1, 1) \mid z_1 \in \mathbb{C} \setminus \{0\}\}$

$E_2 = \{(0, z_2) \mid z_2 \in \mathbb{C} \setminus \{0\}\}$

$E \cong \mathbb{C}^2 \setminus \{0\} / \mathbb{C}^2 \cong \mathbb{C}P^1$

$E_1 \cong [1, z_1]$

$E_2 \cong \mathbb{C}P^1, \quad z_2 \mapsto [z_2, 1]$

$\lambda \cdot (z_1, 1) = (1, z_2) \Rightarrow \quad E_2 = \lambda \times \mathbb{C}P^1_{\lambda \neq 1}$

Change of variables $u \mapsto z \mapsto \lambda z$