Fiber bundles and principal bundles

Def: A surjective smooth map \( \pi: Q \to M \) is a fiber bundle with typical fiber \( F \) if it is locally trivial, i.e.,

\[ \forall x \in M \text{ open nbhd } U \text{ of } x \text{ and a diffeo } U', \pi(U') \to U \times F \]

so that \( \pi'(uv) = uvF \) commutes.

Remarks:
1. Since \( \pi \) is a submersion (d\( \pi \) \( \neq \) 0 auto.), \( \pi \) is a submersion.
2. Any real vector bundle is a fiber bundle with typical fiber a real vector space. Any complex vector bundle is also a fiber bundle.

Example:
\[ M = \mathbb{Q} = \{ \lambda + i \mid \lambda \in \mathbb{R} \} \times \mathbb{R} = \mathbb{R} \times \mathbb{R} \]

Fix \( n \in \mathbb{N} \)

\[ \pi(U(\lambda) - U(1)), \pi(\lambda) = \mathbb{R}^n, \text{ in a fiber bundle with typical fiber } \]

\[ \pi'(\lambda) = \{ \xi \in C \mid \xi^n = \lambda \} \to \mathbb{R}/n. \]

Example:
Klein bottle \( K = \)

in a fiber bundle over \( S^1 \) with typical fiber \( S^1 \)

\[ D/Remark: \text{ Fiber bundles over a fixed manifold } M \text{ form a category. For two fiber bundles } Q_1 \xrightarrow{\pi_1} M, Q_2 \xrightarrow{\pi_2} M \]

\[ \text{Hom}(Q_1, Q_2) = \{ f: Q_1 \to Q_2 \mid f \text{ smooth}, \pi_1 = \pi_2 \text{ commutes} \} \]

A fiber bundle \( Q \to M \) is trivial if it is isomorphic to \( M \times F \xrightarrow{\pi} M \).

Two bundles \( Q_1 \to M, Q_2 \to M \) are isomorphic if there are maps of fiber bundles \( f: Q_1 \to Q_2, g: Q_2 \to Q_1 \) so that \( gf = \text{id}_{Q_1}, fg = \text{id}_{Q_2} \).
The fiber bundle Klein bottle \( S^1 \) is not trivial.

**Def/Ex (Frame bundle)**

Let \( E \to M \) be a real vector bundle of rank \( k \).

We have an open submanifold

\[
\text{Fr}(E) \subseteq \text{Hom}(\mathbb{R}^k \times \mathbb{R}^k, E),
\]

which is a fiber bundle over \( M \) with typical fiber \( \text{Fr}(E)_x = \text{Iso}(\mathbb{R}^k, E_x) \subseteq \text{Hom}(\mathbb{R}^k, E_x) \).

\( \text{GL}(k, \mathbb{R}) \) acts on each fiber of \( \text{Fr}(E) \) on the right

\( \text{Fr}(E)_x \times \text{GL}(k, \mathbb{R}) \to \text{Fr}(E)_x \) \( \cdot A = A \circ f \).

It's not hard to check using local trivializations of \( E \) (hence of \( \text{Fr}(E) \)) that the action is smooth.

**Def** A right action of a Lie group \( G \) on a manifold \( F \) is **free** if \( x \cdot g = x \ \forall x \in F \ \Rightarrow g = e \).

It is **transitive** if \( \forall x, x' \in F \ \exists g \text{ so that } x \cdot g = x' \).

**Ex** Action of \( \text{GL}(k, \mathbb{R}) \) on fibers: \( \text{Fr}(E) \) is a free and transitive:

\( \forall f, f' : \mathbb{R}^k \to E_x \ \exists! A \in \text{GL}(k, \mathbb{R}) \text{ so that } \)

\( f \circ A = f' \).

Indeed we can take \( A = f'^{-1} f \).

**Def** A manifold \( F \) with a free and transitive action of a Lie group \( G \) is called a \( G \)-**torsor**.

**Thus**; fibers of the frame bundle \( \text{Fr}(E) \to M \)

are \( \text{GL}(k, \mathbb{R}) \)-torsors \( (k = \text{rank } E) \)

**Remark** A **section** \( s \) of a fiber bundle \( \pi : E \to M \) is a smooth map \( s : M \to E \) so that \( \pi \circ s = \text{id}_M \).

A local section of \( \pi : E \to M \) is a section of \( \pi|_U : \pi^{-1}(U) \to U \)

for some \( U \subseteq M \) open.

(Local) sections of \( \text{Fr}(E) \to M \) are (local) frames.
Note \( F \) is a \( G \)-torsor so the map
\[
F \times G \rightarrow F \times F, \quad (x, g) \rightarrow (x, x \cdot g)
\]
is a diffeomorphism.

**Def.** (Principal \( G \)-bundle) Let \( G \) be a Lie group. A principal \( G \)-bundle over a manifold \( M \) is a fiber bundle \( P \rightarrow M \) with typical fiber \( G \) and a right action of \( G \) so that local trivializations
\[
\text{Pl}_u = U \times G
\]
are \( G \)-equivariant, i.e.,
\[
\ell(p \cdot g) = \ell(p) \cdot g \quad \text{for all } p \in P, \ g \in G.
\]
Here \( G \) acts on \( U \times G \) by \( (x, a) \cdot g = (x, a\cdot g) \).

**Ex.** For a real vector bundle \( E \rightarrow M \), \( \text{Fr}(E) \rightarrow M \) is a principal \( \text{GL}(k, \mathbb{R}) \)-bundle.

**Remarks.** If \( G \rightarrow P \rightarrow M \) is a principal \( G \)-bundle, then

1. the orbit space \( P/G \) is \( \mathbb{M} \)
2. the fibers \( \pi^{-1}(x) \) of \( P \rightarrow M \) are \( G \)-torsors (fibers are "copies" of \( G \) without the identity)
3. For any manifold \( M \) and any Lie group \( G \), we have the product principal \( G \)-bundle \( M \times G \rightarrow M \).

Principal \( G \)-bundles over a fixed manifold \( M \) form a category \( \text{BG}(M) \) (not entirely standard notation).

A map of principal \( G \)-bundles \( f : P_1 \rightarrow P_2 \) is, by definition, a \( G \)-equivariant map of fiber bundles. Thus,

\[
\begin{array}{ccc}
P_1 & \xrightarrow{f} & P_2 \\
\downarrow \pi_1 & & \downarrow \pi_2 \\
M & \cong & M
\end{array}
\]

and
\[
f(p \cdot g) = f(p) \cdot g \quad \text{for all } p \in P_1, \ g \in G.
\]
Lemma 14.2. A principal $G$-bundle $P \to M$ is trivial ($G$-isomorphic to $M \times G \to M$) if and only if it has a section $s : M \to P$.

Proof $(\Rightarrow)$ Suppose $P \xrightarrow{\Phi} M \times G$ is an $G$-bundle of principal $G$-bundles over $M$. Then $\Phi(x) = (x, \Phi(x))$ is a global section of $P$.

$(\Leftarrow)$ Given a section $s : M \to P$ define $\Phi : M \times G \to P$ by $\Phi(x, g) = s(x) \cdot g$.

Lemma 14.1. Any map $f : P_1 \to P_2$ of principal $G$-bundles over $M$ is an isomorphism.

**WARNING** This does not say that all principal bundles over $M$ are isomorphic to each other. $F : (TS^2) \to S^2$ is not isomorphic to $S^2 \times O(1) : (S^2 \times S^1) \to S^2$.

**Proof** Fix $x_0 \in M$, pick nbd $U$ of $x_0$ so that $P_1|_U, P_2|_U$ have local trivializations $\psi_i : P_i|_U \to U \times G$.

Then $F = \psi_2 \circ f \circ \psi_1^{-1} : U \times G \to U \times G$ is a map of principal $G$-bds, hence has to be of the form $F(x, g) = (x, H(x, g))$ with $H(x, g) = H(x, 1) \cdot g$ for $x \in U$ and $g \in G$.

Let $h(x) = H(x, 1) : U \to G$ since $H \in \mathcal{C}^\infty$. Then $F(x, g) = (x, h(x) \cdot g)$. Hence $(x, g) \mapsto (x, h(x) \cdot g)$ is a $\mathcal{C}^\infty$ inverse of $F$ (since $\sim : G \to G, a \mapsto a^{-1}$ is $\mathcal{C}^\infty$ and since the composition (since $\sim \circ \psi_1^{-1} \circ \psi_2 : U \times G \to U \times G$ is an identity.

\[ (x, g) \mapsto (x, h(x) \cdot g) \mapsto (x, h(x)^{-1} h(x) \cdot g) \mapsto (x, h(x) \cdot (h(x)^{-1} \cdot g)) \]

so $f \circ \psi_1^{-1} = \psi_2 \circ f \circ \psi_1^{-1}$ and $f$ is a diffeo.