Exercise 1. Let $N$ be a manifold, $V$ a finite dimensional vector space and $\alpha \in \Omega^1(N,V)$ a $V$-valued 1-form. Prove that for any vector fields $X,Y$ on $N$ we have

$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]).$$

Exercise 2. Suppose $\pi : Q \to M$ is a surjective submersion. Prove that $\pi^* : \Omega^k(M) \to \Omega^k(Q)$ is injective.

Exercise 3. Recall that for a finite dimensional vector space $V$ the Lie algebra $\mathfrak{gl}(V)$ of the Lie group $GL(V)$ is the vector space $\text{End}(V)$ of all linear maps from $V$ to itself. Prove that given a principal $G$-bundle, $G \to P \xrightarrow{\pi} M$ and a representation $\rho : G \to GL(V)$ the vector bundles

$$\text{End}(P \times^G V) \quad \text{and} \quad P \times^G \mathfrak{gl}(V)$$

are isomorphic. More precisely write down an isomorphism of these bundles. Here, as in the lecture, $G$ acts on $\mathfrak{gl}(V)$ by

$$a \cdot f := \rho(a) \circ f \circ \rho(a^{-1})$$

for all $a \in G$, $f \in \text{End}(V) \equiv \mathfrak{gl}(V)$. 

Homework #5 Math 519
Due in class  Friday, April 6, 2012