MULTILINEAR ALGEBRA NOTES
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1. (Multi)linear algebra

The goal of this note is to define tensors, tensor algebra and Grassmann (exterior) algebra. Unless noted otherwise all vector spaces are over the real number and are finite dimensional. There are two ways to think about tensors:

(1) tensors are multi-linear maps;
(2) tensors are elements of a “tensor product” of two or more vector spaces.

The first way is more concrete. The second is more abstract but also more powerful.

1.1. Tensor products. We start by reviewing multi-linear maps.

Definition 1.1. Let $V_1,\ldots,V_n$ and $U$ be vector spaces. A map $f : V_1 \times \cdots \times V_n \to U$, $(v_1,\ldots,v_n) \mapsto f(v_1,\ldots,v_n)$ is multi-linear if for each fixed index $i$ and a fixed $(n-1)$-tuple of vectors $v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_n$ the map $V_i \to U$, $w \mapsto f(v_1,\ldots,v_{i-1},w,v_{i+1},\ldots,v_n)$ is linear. When the number of factors is $n$, as above, we will also say that $f$ is $n$-linear.

For example, if we identify $\mathbb{R}^{n^2} \simeq \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ by thinking of an $n \times n$ matrix as an $n$-tuple of column vectors, then the determinant

$$\det : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}, \quad (v_1,\ldots,v_n) \mapsto \det(v_1|\ldots|v_n)$$

is an $n$-linear map. Here is an example of a bilinear map. Any inner product on a vector space $V$:

$$V \times V \ni (v,w) \mapsto v \cdot w \in \mathbb{R}$$

is bilinear. There is no standard notation for the space of $n$-linear maps from $V_1 \times \cdots \times V_n$ to $U$. We will denote it by

$$\operatorname{Mult}(V_1 \times \cdots \times V_n, U) = \operatorname{Mult}_n(V_1 \times \cdots \times V_n, U)$$

($n$ is to indicate that these are $n$-linear maps). This space, $\operatorname{Mult}(V_1 \times \cdots \times V_n, U)$, is a vector space: any linear combination of two $n$-linear maps is $n$-linear. We now take a closer look at the space of bilinear maps $\operatorname{Mult}_2(V \times W, U)$. This case is complicated enough to understand what happens with multi-linear maps in general, but simple enough not to bog down in notation.

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Lemma 1.4. Assume that linear maps from a vector space \( X \) to a vector space \( V \) are bilinear and form a basis of \( \text{Mult}_2(V \times W, U) \). Hence
\[
\dim \text{Mult}_2(V \times W, U) = \dim V \dim W \dim U.
\]

Proof. It is easy to see that \( \phi_{ij}^k \) are bilinear. Next, for any \( b \in \text{Mult}_2(V \times W, U) \), any \( w \in W \) and any \( v \in V \),
\[
b(v, w) = b(\sum_{i,j} v_i^*(v)v_i, \sum_{j,k} w_j^*(w)w_j)
= \sum_{i,j,k} v_i^*(v)w_j^*(w)b(v_i, w_j)u_k
= \sum_{i,j,k} v_i^*(v)w_j^*(w)u_k^*(b(v_i, w))u_k
= \sum_{i,j,k} u_k^*(b(v_i, w_j))\phi_{ij}^k(v, w).
\]

Hence the maps \( \phi_{ij}^k \) span \( \text{Mult}_2(V \times W, U) \). Also, the collection of numbers \( u_k^*(b(v_i, w_j)) \) uniquely determine the bilinear form \( b \). Hence \( \phi_{ij}^k \)'s are linearly independent. \( \square \)

We now turn to the definition of the tensor product \( V \otimes W \) [pronounced “\( V \) tensor \( W \)”] of two vector spaces \( V \) and \( W \). Informally it consists of finite linear combinations of symbols \( v \otimes w \), where \( v \in V \) and \( w \in W \). Additionally, these symbols are subject to the following identities:
\[
(v_1 + v_2) \otimes w - v_1 \otimes w - v_2 \otimes w = 0
v \otimes (w_1 + w_2) - v \otimes w_1 - v \otimes w_2 = 0
\alpha (v \otimes w) - (\alpha v) \otimes w = 0
\alpha (v \otimes w) - v \otimes (\alpha w) = 0,
\]
for all \( v, v_1, v_2 \in V, w, w_1, w_2 \in W \) and \( \alpha \in \mathbb{R} \). These identities simply say that the map \( \otimes : V \times W \to V \otimes W \), \( (v, w) \mapsto v \otimes w \), is a bilinear map. The fact that everything in \( V \otimes W \) is a linear combination of symbols \( v \otimes w \) means that the image of the map \( \otimes : V \times W \to V \otimes W \) spans \( V \otimes W \).\(^1\) Here is the formal definition of the tensor product of two vector spaces.

Definition 1.3. A tensor product of two finite dimensional vector spaces \( V \) and \( W \) is a vector space \( V \otimes W \) together with a bilinear map \( \otimes : V \times W \to V \otimes W \), \( (v, w) \mapsto v \otimes w \)\(^2\) such that for any bilinear map \( b : V \times W \to U \) there is a unique linear map \( \tilde{b} : V \otimes W \to U \) with \( \tilde{b}(v \otimes w) = b(v, w) \). That is, the diagram
\[
\begin{array}{ccc}
V \times W & \xrightarrow{b} & U \\
\otimes & \searrow & \\
V \otimes W & \xrightarrow{\tilde{b}} &
\end{array}
\]

commutes. The existence of the map \( \tilde{b} \) satisfying the above conditions is called the universal property of the tensor product.

This definition is quite abstract. It is not clear that such objects exist and, if they exist, that they are unique. Setting the question of existence and uniqueness of tensor products aside, let’s sort out the relationship between \( V \otimes W \) and bilinear maps \( \text{Mult}(V \times W, U) \). Recall that \( \text{Hom}(X, Y) \) is the space of all linear maps from a vector space \( X \) to a vector space \( Y \) and is itself a vector space.

Lemma 1.4. Assume that \( V \otimes W \) exists. Then there is a canonical isomorphism
\[
\text{Hom}(V \otimes W, U) \xrightarrow{\cong} \text{Mult}(V \times W, U).
\]

\(^1\)But the image of \( \otimes \) is not all of \( V \otimes W \). The elements in the image are called decomposable tensors.

\(^2\)The symbol \( v \otimes w \) stands for the value of the map \( \otimes \) on the pair \( (v, w) \).
Proof. The isomorphism in question is built into the definition of the tensor product. Given a linear map $A : V \otimes W \rightarrow U$ the composition $A \circ \otimes : V \times W \rightarrow U$ is bilinear. And conversely, given a bilinear map $b \in \text{Mult}(V \times W; U)$ there is a unique linear map $b : V \otimes W \rightarrow U$ so that $(b \circ \otimes)(v, w) = b(v, w)$ for all $(v, w) \in V \times W$.

In other words the maps $\text{Hom}(V \otimes W, U) \ni A \mapsto A \circ \otimes \in \text{Mult}(V \times W, U)$ and $\text{Mult}(V \times W, U) \ni b \mapsto \tilde{b} \in \text{Hom}(V \otimes W, U)$ are inverses of each other.

Next we observed that the uniqueness of the tensor product is also built into the definition of the tensor product.

**Proposition 1.5.** If tensor products exist, they are unique up to isomorphism.

Proof. The proof is quite formal and uses nothing but the universal property. Suppose there are two vector spaces $V \otimes_1 W$ and $V \otimes_2 W$ with corresponding bilinear maps $\otimes_1 : V \times W \rightarrow V \otimes_1 W$ and $\otimes_2 : V \times W \rightarrow V \otimes_2 W$ which satisfy the conditions of the Definition 1.3. We will argue that these vector spaces are isomorphic. By the universal property there exist a unique linear map $\overline{\otimes}_1 : V \otimes_2 W \rightarrow V \otimes_1 W$ so that the diagram

$$
\begin{array}{ccc}
V \times W & \xrightarrow{\otimes_1} & V \otimes_1 W \\
\downarrow{\otimes_2} & & \downarrow{\overline{\otimes}_1} \\
V \otimes_2 W & & \\
\end{array}
$$

commutes. By the same argument, switching the roles of $\otimes_1$ and $\otimes_2$, there is a unique linear map $\overline{\otimes}_2 : V \otimes_1 W \rightarrow V \otimes_2 W$ making the diagram

$$
\begin{array}{ccc}
V \times W & \xrightarrow{\otimes_2} & V \otimes_2 W \\
\downarrow{\otimes_1} & & \downarrow{\overline{\otimes}_2} \\
V \otimes_1 W & & \\
\end{array}
$$

commute. Define

$$
T_1 = \overline{\otimes}_1 \circ \overline{\otimes}_2 : V \otimes_1 W \rightarrow V \otimes_1 W
$$

$$
T_2 = \overline{\otimes}_2 \circ \overline{\otimes}_1 : V \otimes_2 W \rightarrow V \otimes_2 W.
$$

These are linear maps making the diagrams

$$
\begin{array}{ccc}
V \times W & \xrightarrow{\otimes_1} & V \otimes_1 W \\
\downarrow{T_1} & & \downarrow{\otimes_2} \\
V \otimes_1 W & & V \otimes_2 W
\end{array}
$$

commute. But the identity maps $id_i : V \otimes_i W \rightarrow V \otimes_i W, i = 1, 2,$ are linear and also make the respective diagrams commute. By uniqueness $T_i = id_i$. Hence $\overline{\otimes}_1$ and $\overline{\otimes}_2$ are inverses of each other and provide the desired isomorphisms.

Now we construct the tensor product as a quotient of an infinite dimensional vector space by an infinite dimensional subspace thereby proving its existence.

**Proposition 1.6.** Tensor products exist.

Proof. Let $V$ and $W$ be two finite dimensional vector spaces. We want to construct a new vector space $V \otimes W$ and a bilinear map $\otimes : V \times W \rightarrow V \otimes W$ satisfying the conditions of Definition 1.3. We start with a vector space $F(V \times W)$ made of formal finite linear combinations of ordered pairs $(v, w), v \in V, w \in W$. Its basis is the set $\{(v, w) \mid v \in V, w \in W\} = V \times W$.

If you prefer you can think of $F(V \times W)$ as the set of functions

$$
\{f : V \times W \rightarrow \mathbb{R} \mid f(v, w) \neq 0 \text{ for only finitely many pairs } (v, w)\}.
$$
This set of functions is an infinite dimensional vector space. Its basis consists of functions that take value 1 on a given pair \((v_0, w_0)\) and 0 on all other pairs. It’s tempting to call this function \((v_0, w_0)\).

The vector space \(F(V \times W)\) is called the \textit{free vector space} generated by the set \(V \times W\).

Note that we have an inclusion map \(\iota : V \times W \to F(V \times W)\), \(\iota(v, w) = (v, w)\). It is not bilinear since \((v_1 + v_2, w) \neq (v_1, w) + (v_2, w)\) in \(F(V, W)\).

Consider the map \(\tilde{b} : V \otimes W \to F(V \times W)\) containing the following collection of vectors:

\[
S = \left\{ \frac{(v_1 + v_2, w) - (v_1, w) - (v_2, w)}{v_1, v_2 \in V, \ w} \right\}
\]

In other words, consider the subspace \(K\) of \(F(V \times W)\) spanned by the set \(S\). Define \(V \otimes W\) to be the quotient of \(F(V \times W)\) by \(K\):

\[
V \otimes W := F(V \times W)/K.
\]

Define the map \(\otimes : V \times W \to V \otimes W\) to be the composite of the inclusion \(\iota : V \times W \to F(V \times W)\) and the quotient map \(F(V \times W) \to F(V \times W)/K\). The definition of \(K\) is rigged precisely so that this composite is bilinear. We write \(v \otimes w\) for the value of \(\otimes\) on the pair \((v, w)\). By construction the set \(\{v \otimes w \mid (v, w) \in V \times W\}\) spans \(V \otimes W\) [but it’s much too big to be a basis].

We check that the map \(\otimes : V \times W \to V \otimes W\) has the required universal property. Suppose \(b : V \times W \to U\) is bilinear. Since \(V \times W\) is a basis for \(F(V \times W)\), \(b\) defines a unique linear map \(\tilde{b} : F(V \times W) \to U\) given on the basis by \(\tilde{b}(v, w) = b(v, w)\). As \(b\) is bilinear, \(\tilde{b}\) is 0 on \(K\) by the definition of \(K\). Thus we obtain a linear map \(\tilde{b} : F(V \times W)/K = V \otimes W \to U\) with \(\tilde{b}(v \otimes w) = \tilde{b}(v, w) = b(v, w)\). Since the vectors of the form \(v \otimes w\) span \(V \otimes W\), \(\tilde{b}\) is unique. This verifies the universal property and thereby proves the existence of the tensor product.

\[\textbf{Lemma 1.7.} \quad \text{For any vector spaces } V \text{ and } W, \quad \dim(V \otimes W) = \dim V \cdot \dim W.\]

\[\text{Proof.} \quad \dim V \otimes W = \dim(V \otimes W)^* = \dim \text{Hom}(V \otimes W, \mathbb{R})
\]

\[
= \dim \text{Mult}(V \times W, \mathbb{R}) \quad \text{by Lemma 1.4}
\]

\[
= \dim V \cdot \dim W \cdot \dim \mathbb{R}.
\]

We are now in position to quickly prove a number of results about tensor products.

\[\textbf{Corollary 1.7.1.} \quad \text{If } \{v_i\} \text{ and } \{w_j\} \text{ are a basis of } V \text{ and } W \text{ respectively, then } \{v_i \otimes w_j\} \text{ is a basis of } V \otimes W.\]

\[\text{Proof.} \quad \text{Since the vectors of the form } v \otimes w, \ v \in V, \ w \in W, \text{ span } V \otimes W, \text{ the much smaller set } \{v_i \otimes w_j\} \text{ also spans } V \otimes W. \text{ But } \dim(V \otimes W) = \dim V \cdot \dim W \text{ is precisely the number of elements in the set } \{v_i \otimes w_j\}. \text{ Hence the set } \{v_i \otimes w_j\} \text{ is a basis.} \]

\[\textbf{Lemma 1.8.} \quad V \otimes W \text{ is isomorphic to } W \otimes V.\]

\[\text{Proof.} \quad \text{Consider the map } b : W \times V \to V \otimes W \text{ defined by}
\]

\[b(w, v) = v \otimes w.\]

Since \(b\) is bilinear, there is a unique linear map \(\bar{b} : W \otimes V \to V \otimes W\) with \(\bar{b}(w \otimes v) = v \otimes w\). Since the set \(\{v \otimes w \mid v \in V, w \in W\}\) generates \(V \otimes W\), the map \(\bar{b}\) is surjective. It is an isomorphism by dimension count.

\[\textbf{Lemma 1.9.} \quad V^* \otimes W \text{ is isomorphic to } \text{Hom}(V, W).\]

\[3\text{We are using here the fact that for any } (v, w) \in V \times W, \text{ the tensor } v \otimes w \text{ is a linear combination of } v_i \otimes w_j \text{'s.} \]
Proof. Consider \( b : V^* \times W \to \text{Hom}(V, W) \) defined by
\[
(b(v^*, w))(v) = v^*(v)w \quad \text{for all } v^* \in V^*, v \in V, w \in W.
\]
Since \( b \) is bilinear, it induces a linear map \( \bar{b} : V^* \otimes W \to \text{Hom}(V, W) \) with
\[
(\bar{b}(v^* \otimes w))(v) = v^*(v)w \quad \text{for all } v^* \in V^*, v \in V, w \in W.
\]
Observe that linear maps of the form \( v \mapsto v^*(v)w \) span \( \text{Hom}(V, W) \) (The proof of this fact is very similar to the proof of Lemma 1.2 and is left as an exercise). Hence \( \bar{b} \) is an isomorphism by dimension count. \( \square \)

**Exercise 1.1.** Show that if \( \{v_i\} \) is a basis of a vector space \( V \), \( \{v_i^*\} \) the dual basis and \( \{w_j\} \) the basis of a vector space \( W \), then \( \{v_i^*(\cdot)w_j\} \) is a basis of \( \text{Hom}(V, W) \).

**Lemma 1.10.** If \( A : V \to W \) and \( B : V' \to W' \) are two linear maps, then there is a unique linear map \( A \otimes B : V \otimes V' \to W \otimes W' \) such that \( (A \otimes B)(v \otimes w) = A(v) \otimes B(w) \) for all \( (v, w) \in V \times V' \).

**Proof.** Consider \( b : V \times W \to V' \otimes W' \) given by
\[
b(v, w) = Av \otimes Bw.
\]
The map \( b \) is bilinear, whence the universal property gives us a unique linear map \( \bar{b} : V \otimes W \to V' \otimes W' \) with
\[
\bar{b}(v \otimes w) = Av \otimes Bw
\]
for all \( (v, w) \in V \times W \). \( \square \)

**Exercise 1.2.** Show that if \( A : V \to W \) is represented by a matrix \( (a_{ij}) \) with respect to some bases of \( V \) and \( W \) and \( B : V' \to W' \) is represented by a matrix \( (b_{kl}) \) with respect to bases of \( V' \) and \( W' \), then \( A \otimes B \) is represented by the matrix \( (a_{ij}b_{kl}) \) with respect to the appropriate bases.

**Exercise 1.3.** Show that there is a natural isomorphism \( \phi : V^* \otimes W^* \cong \text{Mult}(V \times W, \mathbb{R}) \) with
\[
\phi(v^* \otimes w^*)(v, w) = v^*(v)w^*(w)
\]
for all \( v^*, w^*, v, w \).

Show that there is a natural isomorphism \( \psi : V^* \otimes W^* \to (V \otimes W)^* \) with
\[
\psi(v^* \otimes w^*)(v \otimes w) = v^*(v)w^*(w)
\]
for all \( v^*, w^*, v, w \).

**Exercise 1.4.** Show that the map \( \mathbb{R} \times V \to V, (a, v) \mapsto av \) gives rise to an isomorphism \( \mathbb{R} \otimes V \cong V \) which sends \( a \otimes v \) to \( av \) for all \( a \in \mathbb{R} \) and \( v \in V \).

**Exercise 1.5.** Show that taking tensor product is associative:
\[
V \otimes (U \otimes W) \cong (V \otimes U) \otimes W
\]
for any three vector spaces \( V, U \) and \( W \).

From now on we write \( V \otimes U \otimes W \) for \( V \otimes (U \otimes W) \) since the order of taking tensor products doesn’t matter. Exercise 1.5 above also allows us to define recursively tensor powers of a vector space \( V \). We define
\[
V^\otimes 0 := \mathbb{R}, \quad V^\otimes 1 := V \quad \text{and} \quad V^\otimes n := V^\otimes (n-1) \otimes V \quad \text{for } n > 1.
\]

It is not hard to generalize the relationship between bilinear maps and tensor products to the relationship between \( n \)-linear maps and \( n \)-fold tensor products. For example:
Exercise 1.6. Prove that given a $n$-linear map

$$f : V \times \cdots \times V \to U,$$

then there exists a unique linear map $\overline{f} : V^\otimes n \to U$ with

$$\overline{f}(v_1 \otimes \cdots \otimes v_n) = f(v_1, \ldots, v_n).$$

for all $(v_1, \ldots, v_n) \in V \times \cdots \times V$.

Moreover, given $a \in V^\otimes n$ and $b \in V^\otimes m$, $a \otimes b$ is in $V^\otimes n \otimes V^\otimes m \simeq V^\otimes (n+m)$. This gives us an $\mathbb{R}$-bilinear map,

$$V^\otimes n \times V^\otimes m \to V^\otimes (n+m), \quad (a, b) \mapsto a \otimes b.$$

Note that if $n = 0$ the map above is simply $\mathbb{R} \times V^\otimes m \to V^\otimes m$, $(a, t) \mapsto at$.

(cf. Exercise 1.4).

Definition 1.11. An algebra over $\mathbb{R}$ is a vector space $A$ together with a bilinear map $A \times A \to A$, $(a, a') \mapsto aa'$ ("multiplication"). An algebra $A$ is said to be an algebra with unity if there is an element $1 \in A$ such that $1 \cdot a = a$ for all $a \in A$. An algebra $A$ is associative if the multiplication is associative.

Remark 1.12. Note that in any algebra $A$, $0a = a0 = 0$ for all $a \in A$ (this is because multiplication is required to be bilinear).

Remark 1.13. If $A$ is an algebra with 1 then there is an injection $\mathbb{R} \to A$, $x \mapsto x1$. We will always identify $\mathbb{R}$ with its image in $A$.

Example 1.14. A Lie algebra is an algebra. It is not associative and does not have 1 (why not?).

Example 1.15. The space $M_n(\mathbb{R})$ of $n \times n$ matrices forms an algebra under matrix multiplication. It is an algebra with unity: the identity matrix $I$ is the unity.

Definition 1.16. An algebra $A$ is graded if

$$A = \sum_{i=0}^\infty A_i$$

direct sum

and if for any $a \in A_i$ and $b \in A_j$ we have $a \cdot b \in A_{i+j}$. We will refer to the elements of $A_k$ as elements of degree $k$.

Given a vector space $V$ we construct the corresponding tensor algebra $T(V)$ as follows. As a vector space $T(V)$ is the direct sum:

$$T(V) = \mathbb{R} \oplus V \oplus V^\otimes 2 \oplus \cdots \oplus V^\otimes n \oplus \cdots = \sum_{i=0}^\infty V^\otimes i.$$ 

Thus the elements of $T(V)$ are finite sums $a_{i_1} + a_{i_2} + \cdots + a_{i_k}$, $a_{i_j} \in V^\otimes i_j$. We define the multiplication on $T$ by extending the multiplication

$$V^\otimes n \times V^\otimes m \to V^\otimes (n+m), \quad (a, b) \mapsto a \otimes b.$$

bilinearly to all of $T(V)$. The tensor algebra $T(V)$ of a vector space $V$ is a graded associative algebra with 1. Note that by construction the elements of $T(V)$ are sums of products of elements of $V$, that is, $T(V)$ is generated by $V$.
1.2. The Grassmann (exterior) algebra and alternating maps. We have seen that tensor products are intimately related to multi-linear maps. Exterior (Grassmann) algebras are just as intimately related to alternating multi-linear maps. Recall that an n-linear map \( f : V \times \cdots \times V \to U \) is alternating if it changes sign whenever we switch to adjacent entries:

\[
f(v_1, \ldots, v_i, v_{i+1}, \ldots, v_n) = -f(v_1, \ldots, v_{i+1}, v_i, \ldots, v_n)
\]

for all \((v_1, \ldots, v_n) \in V \times \cdots \times V\) and any index \(i\).

**Example 1.17.** The determinant

\[
\det: \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}, \quad (v_1, \ldots, v_n) \mapsto \det(v_1|\ldots|v_n)
\]

is an alternating map.

**Example 1.18.** Consider a vector space \( V \) and \( a, b \in V^* \). Define the bilinear map \( a \wedge b \) by

\[
(a \wedge b)(v_1, v_2) := a(v_1)b(v_2) - a(v_2)b(v_1), \quad v_1, v_2 \in V.
\]

The map \( a \wedge b \) (“a wedge b”) is alternating.

**Definition 1.19** (Grassmann (exterior) algebra). Let \( V \) be a finite dimensional vector space over \( \mathbb{R} \). The Grassmann (exterior) algebra \( \Lambda^*(V) \) is an algebra over \( \mathbb{R} \) with unity together with an injective linear map \( i: V \to \Lambda^*(V) \) called the structure map which has the following universal property: If \( A \) is an algebra over \( \mathbb{R} \) with unity and \( j: V \to A \) is a linear map such that \( j(v) \cdot j(v) = 0 \) for all \( v \in V \), then there is a unique algebra map \( \eta: \Lambda^*(V) \to A \) such that the following diagram commutes:

\[
\begin{array}{ccc}
V & \xrightarrow{i} & \Lambda^*(V) \\
\downarrow & & \downarrow \eta \\
A & \xrightarrow{j} & A
\end{array}
\]

**Proposition 1.20.** If the exterior algebra \( \Lambda^*(V) \) exists, it is unique (up to isomorphism).

**Proof.** This is a formal exercise and is left to the reader. \( \square \)

**Proposition 1.21.** For every vector space \( V \) the exterior algebra \( \Lambda^*(V) \) exists.

**Proof.** Let \( I \) be the two-sided ideal in the tensor algebra \( T(V) \) generated by the set \( \{v \otimes v : v \in V\} \). Note that \( \mathbb{R} \cap I = 0 \) and \( V \cap I = 0 \) for degree reasons. Define

\[
\Lambda^*(V) := T(V)/I,
\]

the quotient of the tensor algebra by the ideal \( I \). Then \( \Lambda^*(V) \) is an algebra — it inherits the multiplication from \( T(V) \). The induced multiplication in \( \Lambda^*(V) \) is denoted by \( \wedge \) (“wedge”). Since the tensor algebra is graded, so is \( I \), and

\[
I = (I \cap V \otimes 2) \oplus (I \cap V \otimes 3) \oplus \cdots
\]

Since \( V \cap I = 0 \), the composite \( i: V \to T(V) \to T(V)/I = \Lambda^*(V) \) is an injection. Note that any element of \( \Lambda^*(V) \) is a finite linear combination of products of elements of \( V \).

Now that we have constructed the exterior \( \Lambda^*(V) \), let us prove the universal property. Suppose that \( A \) is an algebra and that we are given a linear map \( j: V \to A \) with \( j(v) \cdot j(v) = 0 \) for all \( v \in V \). Consider the map \( b: V \times V \to A \) given by \( b(v, w) = j(v) \cdot j(w) \). Since \( b \) is bilinear, there is a unique linear map \( j^{(2)}: V \otimes V \to A \) with \( j^{(2)}(v \otimes w) = j(v) \cdot j(w) \). Similarly, for all positive integers \( k \), we have \( k \)-linear maps \( j^{(k)}: V \otimes k \to A \) with

\[
j^{(k)}(v_1 \otimes \cdots \otimes v_k) = j(v_1) \cdots j(v_k).
\]

In addition, we define \( j^{(0)}(a) = a \cdot 1_A \), for all \( a \in \mathbb{R} \). In this way, we obtain an algebra map \( \overline{j}: T(V) \to A \).

By assumption, \( \overline{j}(v \otimes v) = 0 \) for all \( v \in V \). Therefore \( \overline{j} \) vanishes on the ideal \( I \). This implies that \( \overline{j} \) descends to an algebra map \( \eta: \Lambda^*(V) = T/I \to A \) with \( \eta(v) = j(v) \) for all \( v \in V \). Since an algebra map is uniquely determined on generators, and since \( V \) generated \( \Lambda^*(V) \), the map \( \eta \) is unique. \( \square \)

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4 A map \( f: A \to B \) between two algebras is an algebra map if \( f \) is linear and preserves multiplication: \( f(a_1a_2) = f(a_1)f(a_2) \).
Remark 1.22. For any $v \in V$, we have $v \wedge v = 0$ in the exterior algebra $\Lambda^*(V)$. Also, 
\[ 0 = (v_1 + v_2) \wedge (v_1 + v_2) = v_1 \wedge v_1 + v_1 \wedge v_2 + v_2 \wedge v_1 + v_2 \wedge v_2 \]
gives that 
\[ v_1 \wedge v_2 = -v_2 \wedge v_1; \]
That is, the wedge product is skew-commutative.

Remark 1.23. Let $\Lambda^k(V) = T^k(V)/(T^k(V) \cap I)$. The vector space $\Lambda^k(V)$ is called the $k$th exterior power of $V$. Then 
\[ \Lambda^*(V) = \sum_{k=0}^{\infty} \Lambda^k(V), \]
where 
\[ \Lambda^0(V) = \mathbb{R} \quad \text{and} \quad \Lambda^1(V) = V. \]
Also, if $\alpha \in \Lambda^k(V)$ and $\beta \in \Lambda^l(V)$, then $\alpha \wedge \beta \in \Lambda^{k+l}(V)$. Thus, $\Lambda^*(V)$ is a graded algebra with $1$.

Remark 1.24. We know that if $\{v_1, \ldots, v_n\}$ is a basis for $V$, then $\{v_i \otimes v_j\}$ is a basis for $V \otimes V$. By induction, $\{v_i \otimes \cdots \otimes v_k\}$ is a basis for $V^{\otimes k}$. Thus, $\{v_{i_1} \wedge \cdots \wedge v_{i_k}\}$ generates $\Lambda^k(V) = V^{\otimes k}/(I \cap V^{\otimes k})$. Since $\wedge$ is skew-commutative, however, we can reduce this generating set to a smaller one: 
\[ \{v_{i_1} \wedge \cdots \wedge v_{i_k} \mid i_1 < \cdots < i_k\}, \]
This implies that 
\[ \Lambda^l(V) = 0 \quad \text{whenever} \quad l > \dim V. \]
We will see below that the set (1.1) is a basis of $\Lambda^k(V)$.

We now investigate the connection between the $k$-th exterior power $\Lambda^k(V)$ of a vector space $V$ and alternating maps.

Proposition 1.25 (Universal property of $k$-th exterior power of a vector space). Let $U$ and $V$ be vector spaces. If $f : \bigwedge^k U \to V$ is alternating then there is a unique linear map $\overline{f} : \Lambda^k(V) \to U$ with 
\[ \overline{f}(v_1 \wedge \cdots \wedge v_k) = f(v_1, \ldots, v_k). \]
Proof. By the universal property of $V^{\otimes k}$, there is a unique linear map $\tilde{f} : V^{\otimes k} \to U$ such that $\tilde{f}(v_1 \otimes \cdots \otimes v_k) = f(v_1, \ldots, v_k)$. Since $f$ is alternating, $\tilde{f}\mid_{I \cap V^{\otimes k}} = 0$, where $I$ is the ideal defined in the construction of $\Lambda^*(V)$. This gives us the linear map $\overline{f} : \Lambda^k(V) = V^{\otimes k}/(I \cap V^{\otimes k}) \to U$ with the desired property. \qed

Corollary 1.25.1. The space of $k$-linear alternating maps $\{f : \bigwedge^k U \to V \mid f \text{ is alternating}\}$ is isomorphic to the space $\text{Hom}(\Lambda^k(V), U)$.

Lemma 1.26. Let $V$ be an $n$-dimensional vector space. Then $\Lambda^n(V)$ is 1-dimensional.
Proof. We may assume that $V = \mathbb{R}^n$. Let $e_1, \ldots, e_n$ be the standard basis. Then $e_1 \wedge \cdots \wedge e_n$ spans $\Lambda^n(V)$. We need to show that $e_1 \wedge \cdots \wedge e_n \neq 0$. The determinant $\det : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ is 1 on the identity matrix $I = (e_1 \cdots e_n)$: $\det(e_1 \cdots e_n) = 1$. Hence the induced linear map $\overline{\det} : \Lambda^n(\mathbb{R}^n) \to \mathbb{R}$ is 1 on $e_1 \wedge \cdots \wedge e_n$. Therefore $e_1 \wedge \cdots \wedge e_n \neq 0$. \qed

Corollary 1.26.1. If $\{f_1, \ldots, f_n\}$ is a basis for a vector space $V$, then $\{f_{i_1} \wedge \cdots \wedge f_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}$ is a basis for its $k$-th exterior power $\Lambda^k(V)$.
Proof. By Remark 1.24 the above set generates $\Lambda^k(V)$. So we only need to check independence. Suppose 
\[ 0 = \sum_{i_1 < \cdots < i_k} a_{i_1, \ldots, i_k} f_{i_1} \wedge \cdots \wedge f_{i_k} \]
for some $a_{i_1, \ldots, i_k} \in \mathbb{R}$.
Pick a sequence $j_1 < j_2 < \cdots < j_k$. Let $j_{k+1} < \cdots < j_n$ be the remaining indices. Then 
\[ \sum_{i_1 < \cdots < i_k} a_{i_1, \ldots, i_k} f_{i_1} \wedge \cdots \wedge f_{i_k} = a_{j_1, \ldots, j_k} f_{j_1} \wedge \cdots \wedge f_{j_k} \wedge f_{j_{k+1}} \wedge \cdots \wedge f_{j_n}, \]
since \( a_{i_1, \ldots, i_k} f_{i_1} \wedge \cdots \wedge f_{i_k} = 0 \) whenever \( i_s = j_r \) for some \( s, r \). This gives \( a_{j_1, \ldots, j_k} = 0 \). Also, 
\( f_{j_1} \wedge \cdots \wedge f_{j_k} \wedge f_{j_{k+1}} \wedge \cdots \wedge f_{j_n} = \pm f_1 \wedge \cdots \wedge f_n \neq 0 \). Hence \( f_{j_1} \wedge \cdots \wedge f_{j_k} \neq 0 \). □

**Corollary 1.26.2.** For any finite dimensional vector space \( V \)
\[
\dim \Lambda^k(V) = \binom{\dim V}{k} = \frac{(\dim V)!}{k!(\dim V - k)!}.
\]

**Lemma 1.27.** Let \( A : V \to W \) be a linear map. Then there is a unique linear map \( \Lambda^k(A) : \Lambda^k(V) \to \Lambda^k(W) \) such that 
\[
(\Lambda^k(A))(v_1 \wedge \cdots \wedge v_k) = Av_1 \wedge \cdots \wedge Av_k
\]
for all \( v_1, \ldots, v_k \in V \).

**Proof.** Consider the map \( b : V \times \cdots \times V \to \Lambda^k(W) \) given by 
\[
b(v_1, \ldots, v_k) = Av_1 \wedge \cdots \wedge Av_k.
\]
Since \( \wedge \) is skew-commutative, \( b \) is an alternating map. By Proposition 1.25 there exists a unique linear map \( \Lambda^k(A) : \Lambda^k(V) \to \Lambda^k(W) \) with the required properties. □

**Exercise 1.7.** Let \( A : V \to W \) be a linear map as above. Choose bases of \( V \) and \( W \) and the corresponding bases of \( \Lambda^k(V) \) and of \( \Lambda^k(W) \). Show that the entries of the matrix representing \( \Lambda^k(A) \) are polynomial in the entries of the matrix representing \( A \).

1.3. **Pairings.**

**Definition 1.28.** Let \( V \) and \( W \) be two vector spaces. A **pairing** is a bilinear map \( \langle \cdot, \cdot \rangle : V \times W \to \mathbb{R} \).

**Example 1.29.** Let \( V \) be a vector space and \( V^* \) be its dual. The evaluation map 
\[
V^* \times V \to \mathbb{R} \quad \langle \ell, v \rangle = \ell(v)
\]
is a pairing.

**Definition 1.30.** A pairing \( \langle \cdot, \cdot \rangle : V \times W \to \mathbb{R} \) is **non-degenerate** if
\[
\langle v_0, w \rangle = 0 \quad \forall w \in W \Rightarrow v_0 = 0
\]
\[
\langle v, w_0 \rangle = 0 \quad \forall v \in V \Rightarrow w_0 = 0.
\]

**Example 1.31.** The evaluation map 
\[
V^* \times V \to \mathbb{R} \quad \langle \ell, v \rangle = \ell(v)
\]
is a non-degenerate pairing. In a sense it is the only nondegenerate pairing:

**Proposition 1.32.** If \( b : V \times W \to \mathbb{R} \) is a nondegenerate pairing, then \( V \simeq W^* \) and \( W \simeq V^* \).

**Proof.** Consider \( b^\#_1 : V \to W^* \) given by 
\[
(b^\#_1(v))(w) = b(v, w).
\]
The map \( b^\#_1 \) is linear, and
\[
\ker b^\#_1 = \{ v_0 \in V : b^\#_1(v_0) = 0 \} = \{ v_0 \in V : b(v_0, w) = 0 \, \forall w \} = \{0\}.
\]
Thus \( \dim V \leq \dim W^* = \dim W \). By the same argument, we have \( \dim W \leq \dim V^* = \dim V \). Therefore \( \dim V = \dim W \). Hence \( b^\#_1 \) is an isomorphism.

By the same argument, \( b^\#_2 : W \to V^* \) given by \( w \mapsto b(\cdot, w) \) is an isomorphism as well. □

**Proposition 1.33.** There is a nondegenerate pairing 
\[
\langle \cdot, \cdot \rangle : \Lambda^k(V^*) \times \Lambda^k(V) \to \mathbb{R}
\]
with 
\[
(v_1^* \wedge \cdots \wedge v_k^*, v_1 \wedge \cdots \wedge v_k) = \det \left( v_i^*(v_j) \right).
\]
Hence 
\[
\Lambda^k(V^*) \simeq (\Lambda^k(V))^*.
\]
Proof. Consider \( b : V^* \times \cdots \times V^* \times V \times \cdots \times V \rightarrow \mathbb{R} \) given by
\[
b(l_1, \ldots, l_k, v_1, \ldots, v_k) = \det \left( l_i(v_j) \right).
\]
For a fixed \((l_1, \ldots, l_k) \in V^* \times \cdots \times V^*\), \( b \) is alternating in the \( v \)'s. So there is a map \( \overline{b} : (V^* \times \cdots \times V^*) \times \Lambda^k(V) \rightarrow \mathbb{R} \) with
\[
(l_1, \ldots, l_k, v_1 \wedge \cdots \wedge v_k) \mapsto \det \left( l_i(v_j) \right).
\]
Similarly, for a fixed \( v_1 \wedge \cdots \wedge v_k \in \Lambda^k(V)\), \( \overline{b} \) is alternating in the \( l \)'s, which means that there is a map \( \tilde{b} : \Lambda^k(V^*) \times \Lambda^k(V) \rightarrow \mathbb{R} \) with the desired property.
To check non-degeneracy evaluate the pairing on the respective bases. \( \square \)

Combining the proposition above with Corollary 1.25.1 we get:

**Corollary 1.33.1.** The space of \( k \)-linear alternating maps \( \{ f : V \times \cdots \times V \rightarrow \mathbb{R} \mid f \text{ is alternating} \} \) is isomorphic to the \( k \)-th exterior power \( \Lambda^k(V^*) \).

**Remark 1.34.** Explicitly \( \ell_1 \wedge \cdots \wedge \ell_k \in \Lambda^k(V^*) \) defines a \( k \)-linear alternating map by
\[
\ell_1 \wedge \cdots \wedge \ell_k (v_1, \ldots, v_k) = \det(\ell_i(v_j))
\]
for all \( v_1, \ldots, v_k \in V \). In particular
\[
\ell_1 \wedge \ell_2 (v_1, v_2) = \ell_1(v_1)\ell_2(v_2) - \ell_1(v_2)\ell_2(v_1)
\]

**Exercise 1.8.** Suppose that \( V \) is an \( n \)-dimensional vector space. Given a linear map \( A : V \rightarrow V \), we get a map \( \Lambda^n(A) : \Lambda^n(V) \rightarrow \Lambda^n(V) \), and since \( \dim \Lambda^n(V) = 1 \), the map \( \Lambda^n(A) \) is multiplication by a scalar. Show that this scalar is \( \det A \).