Recall: Want to show: \( M \) connected, orientable manifold of dimension \( m \).

Then \( \tilde{f}_m: H^*_c(M) \to \mathbb{R} \), \( \tilde{f}_m(\omega) = \tilde{f}_m(\omega') \) in an iso.

We have proved so far:

1. \( H^*_c(U) \cong \mathbb{R} \), for any \( U \) diffeomorphic to \( \mathbb{R}^m \).

In particular, for any \( U \) diffeomorphic to a ball \( B_\epsilon(0) \subseteq \mathbb{R}^m \).

2. If \( \omega \in \Omega^*_c(M) \), then \( \exists \) open sets \( V \subseteq V_k \) diffeomorphic to balls and \( \omega \in \Omega^*_c(V_k) \) st: \( \omega = \omega + \omega_k \).

Want to show:

Fix \( U \subseteq M \) open, \( U \) diffeomorphic to a ball.

Then the open embedding \( U \subseteq M \) induces an iso \( H^*_c(U) \cong H^*_c(M) \).

By (2) enough to show:

3. If \( \omega \in \Omega^*_c(U) \) with \( \text{supp}(\omega) \subseteq V \), \( V \) diffeomorphic to a ball, then \( \exists \eta \in \Omega^{-1}_c(U) \), \( \omega' \in \Omega^*_c(U) \) so that \( \omega = \omega' + d\eta \).

Now (3) says: \( \tilde{f}_m: H^*_c(U) \to H^*_c(M) \) is onto. We know \( \tilde{f}_m: H^*_c(U) \to \mathbb{R} \) is onto, \( \tilde{f}_m \circ \tilde{f}_m = \tilde{f}_m \) commute,

and \( \tilde{f}_m: H^*_c(U) \to \mathbb{R} \) is an iso.

4. Suppose now we are in case (3): \( \forall \omega \in \Omega^*_c(U) \), \( d\eta \omega = 0 \) then, since \( H^*_c(U) \cong \mathbb{R} \)

\[ \ker(d\eta \circ \omega) = d\Omega^{-1}_c(U) \] \( \Rightarrow \omega = d\eta \) for some \( \eta \in \Omega^{-1}_c(U) \).

\[ \Rightarrow \omega = 0 = d\eta \in \Omega^{-1}_c(U) \text{ with } 0 \in \Omega^{-1}_c(U) \).

5. Suppose now \( \tilde{f}_m(\omega) \neq 0 \), and \( U \cap V \neq \emptyset \). Pick \( \omega \in U \cap V \) so that \( W \subseteq B_\epsilon(0) \) for some \( R \). Choose \( f \in C^\infty_c(B_\epsilon(0)) \) with \( f|_W = 1 \) \( \Rightarrow \omega' = \tilde{f}_m(\omega) \) is an iso, \( \exists \eta \in \Omega^{-1}_c(U) \) such that \( \omega - \omega' = d\eta \).

We have: \( f \) \( \text{supp} \omega \subseteq W \subseteq V \cap U \), \( \tilde{f}_m(\omega) = \tilde{f}_m(\omega') \).

Since \( \tilde{f}_m: H^*_c(U) \to \mathbb{R} \) is an iso, \( \exists \eta \in \Omega^{-1}_c(U) \) st: \( \omega - \omega' = d\eta \).
By previous discussion. This completes the proof.

\[ \lim_{n \to \infty} \lambda_n(x) = \lambda(x) \]

Next, consider the case where \( V \cap U = \emptyset \).

Given \( \omega \in \partial\mathcal{D}(M) \) with \( \omega - \omega(0) = d_\nu \),

and \( \gamma \in \gamma_{x} \).

(iii) \( W \subseteq V \) with \( \mathcal{W} = W \).

W. \ n \ W \neq \emptyset.

\( \exists \omega(0) \in \mathcal{O}_n(\mathcal{M}) \).

for all \( \delta \in \partial\mathcal{D}(M) \).

by finitely many open sets \( \mathcal{W}_0, \mathcal{W}_1, \ldots, \mathcal{W}_t \) such that \( \gamma \cap \mathcal{D}(M) \) is compact, we can cover it.

Finally, consider the case where \( V \cap U = \emptyset \).

\[ \sup_{u \in \mathcal{U}} \sum_{i=1}^{n} \| u_i - u \| = d \] on \( M \).
First application: degree of a map.

Let \( M, N \) be two connected oriented manifolds of dimension \( n \) and \( f: N \rightarrow M \) in \( C^\infty \), and \((proper?)\). We have isomorphisms

\[
\begin{align*}
&\tilde{s}_N : H^n_c(N) \xrightarrow{\sim} \mathbb{R}^n, \\
&\tilde{s}_M : H^n_c(M) \xrightarrow{\sim} \mathbb{R}^n.
\end{align*}
\]

and a map

\[
H^n_c(f) \equiv f^* : H^n_c(N) \rightarrow H^n_c(N), \quad \omega \mapsto f^*(\omega).
\]

This gives us a linear map

\[
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{f^*} & H^n_c(N) \\
(\tilde{s}_M)^{-1} & \downarrow & \downarrow \\
& & \tilde{s}_N
\end{array}
\]

Define:

\[
\text{deg}(f) := \text{image of } 1 \text{ under this map (degree of } f).\]

It's defined by:

- Pick \( \omega \in \mathcal{C}_c^\infty (M) \) with \( \tilde{s}_M(\omega) = 1 \)
- \( \text{deg } f := \tilde{s}_N(f^*(\omega)) \).

**Remarks:**

1. If \( N = M \), then \( \text{deg}(f) \) doesn't depend on a choice of orientation of \( M \): changing orientation of \( M \) changes sign in \( \tilde{s}_M : H^n_c(M) \rightarrow \mathbb{R}^n \).

Since

\[
\text{deg } f = \left[ (\tilde{s}_M \circ f^* \circ (\tilde{s}_M)^{-1}) \right](1),
\]

\( \text{deg } f \) is unaffected.

2. Suppose \( M \) and \( N \) are compact. Then
   \begin{enumerate}
   \item any map \( f: M \rightarrow N \) is proper
   \item \( H^\ast_c(M) = H^\ast(M) \)
   \end{enumerate}

Hence \( \forall f : M \rightarrow N \)

\[
\text{deg}(f) \text{ depends only on } H^n(f) : H^n(M) \rightarrow H^n(M),
\]

which depends only on the homotopy class of \( f \).
Remarks Suppose \( U, V \subseteq \mathbb{R}^n \) are open sets diffeomorphic \( \sim \), to open balls and
\[
f: U \to V \text{ a diffeo. (hence proper)}
\]

Claim \( \deg f = \left\{ \begin{array}{ll} +1 & \text{f preserves orientation} \\ -1 & \text{f reverses orientation} \end{array} \right. \)

Proof \( \forall \psi \in C_c^\infty (V) \), change of variables formula says:
\[
\int_U \psi(x) dx = \int_V \psi(f(y)) \left| \det Df(y) \right| dy
\]
\[
= \left\{ \begin{array}{ll} \int_U f^* (\psi(x) dx, v - adx_v), \det Df > 0 \\ - \int_U f^* (\psi(x) dx_v - adx_v), \det Df < 0 \end{array} \right.
\]

Consequence of remark 3

Suppose \( M, N \) compact connected oriented and \( f: N \to M \) is smooth. Then the degree \( \deg f \) is an integer. Moreover, it has the following description:

Let \( q \in M \) be a regular value of \( f \). Then \( f^{-1}(q) \) is a finite set of points and

\[
\deg f = \# \{ p \in f^{-1}(q) \mid Df(p) \text{ preserves orientation} \}
\]

\[ - \# \{ p \in f^{-1}(q) \mid Df(p) \text{ reverses orientation} \} \]

Reason: Since \( q \) is a regular value, \( f^{-1}(q) \) is a closed submanifold of \( N \) with \( \dim f^{-1}(q) = \dim M - \dim N = 0 \), hence \( f^{-1}(q) \) has a finite set of points. Since \( q \) is a regular value \( \forall p \in f^{-1}(q) \) \( U_p \) is an \( f \)-preimage.

May assume \( U_p \cap U_{p'} = \emptyset \) for \( p \neq p' \).

Now choose \( W \subseteq \bigcap_{p \in f^{-1}(q)} U_p \) with \( \sum W = 1 \).

Now compute \( \int_W f^* \omega \).