Last time: Given a connection \( \alpha \) on a principal \( G \)-bundle, we constructed its curvature \( F \in \Omega^2(M, P \times^gG) \).

We now try to sort this out when \( G = S^1 = \mathbb{R}/\mathbb{Z} \).

**Aside: Basic forms on principal \( G \)-bundles.**

Given \( G \to P \xrightarrow{\pi} M \), we have a map \( \pi^* : \Omega^k(M) \to \Omega^k(P) \) with \( \pi^* \) \( \pi \)-injective:

for \( \omega \in \Omega^k(M) \) with \( \pi^* \omega \), \( (\pi^* \omega)_p = 0 \) \( \implies \) \( \forall v_1, \ldots, v_k \in T_pP \)

\[
0 = (\pi^* \omega)_p(v_1, \ldots, v_k) = \omega_{\pi(p)}(d\pi(v_1), \ldots, d\pi(v_k))
\]

Since \( d\pi \) is onto, \( \omega_{\pi(p)}(v_1, \ldots, v_k) = 0 \) \( \forall v_1, \ldots, v_k \in T_{\pi(p)}M \)

Since \( \pi \) is onto, \( \omega_p = 0 \) \( \forall \pi(M) = \omega = 0. \)

**Proposition 35.1**

\[
(\gamma) \pi^*(\Omega^k(M)) = \{ \gamma + \Omega^k(P) \} \quad \text{for } \gamma \in \Omega^k(P) \text{ and } v \in \mathbb{P}P.
\]

**Remarks:**

1. RHS \( \gamma \) is called the space of basic forms, and is denoted by \( \Omega^k(P) \) \text{basic}.  
2. If \( \Phi : P \to M \) is a principal \( \text{SL}^1 \)-bundle and \( \alpha \in \Omega^1(P, \mathbb{R}) \) a connection 1-form, then it's not hard to check: \( d\alpha \) is basic.

Hence \( d\alpha = \pi^* F \) for some \( F \in \Omega^2(M) \)

We'll show: \( F = \text{curvature of } \alpha \).

1. Also, since \( 0 = d(d\alpha) = d(\pi^* F) - \pi^* (dF) \) and \( \pi^* \) is injective, \( dF = 0 \) \( \implies \) \( F \) defines a class \( [F] \in H^2(M, \mathbb{R}) \) called the 1st Chern class of \( \Phi : P \to M \).

We'll show: \( [F] \) does not depend on the choice of the connection \( \alpha \in \Omega^1(P, \mathbb{R}) \).
Proof of proposition. Since $\forall a \in G \quad \nabla^a_{\rho a} \circ R_{\rho a} = 0$, $\forall w \in \Omega^k(M)$, $\pi^* w = (\pi^* R_{\rho a})^* w = R_{\rho a}^* (\pi^* w)$.

$\forall p \in P, \forall u \in T_p P$, $d\pi_p (u) = 0$.

$\Rightarrow \forall u_1, \ldots, u_{k-1} \in T_p P, \forall w \in \Omega^k(M)$

$\left( \pi^* (w) \right)_p (u_1, \ldots, u_{k-1}) = \pi^* (w) (u, u_1, u_{k-1})$

$= W_{\nabla^a p} \left( \frac{d\pi_p u}{\nabla^a}, \frac{d\pi_p u_1}{\nabla^a}, \ldots, \frac{d\pi_p u_{k-1}}{\nabla^a} \right) = 0.$

$\Rightarrow \pi^* \Omega^k(M) \subseteq \Omega^k(P)$ basic.

Conversely, suppose $\tau \in \Omega^k(P)$ basic. Given $x \in M$, define $W_x \in \Lambda^k(T_x M)$ by

$W_x (w_1, \ldots, w_k) = T_p (u_1, \ldots, u_k)$

where $p \in \pi^*(x)$ and $u_1, \ldots, u_k \in T_p P$ with $d\pi_p (u_i) = w_i$.

We argue that $W$ is a well-defined smooth $k$-form on $M$ with $\pi^* W = \tau$.

- choice of $u_i$'s doesn't matter: say $u'_i \in T_p P$ is another vector with $d\pi_p (u'_i) = w_i$. Then $d\pi_p (u_i - u'_i) = 0$.

$\Rightarrow u = u_i - u'_i$ is vertical. $\Rightarrow$

$T_p (u_1, u_2, \ldots, u_k) = T_p (u_1', u_2, \ldots, u_k) + T_p (u_1', u_2, \ldots, u_k)$

$= 0 + T_p (u_1', u_2, \ldots, u_k),$

- choice of $p$ doesn't matter: suppose $p' \in \pi^*(x)$.

Then $p' = p \cdot a$ for some $a \in G$, so let $u_i$'s be as above:

$u_i \in T_p P$ with $d\pi_p (u_i) = w_i$. Then $d\pi_p \cdot ((dR_a)_{\pi^*} (u_i) = d(\pi^* R_{\rho a})_{\pi^*} (u_i) = d \pi_p (u_i) = w_i$ and

$T_p' \left( (dR_a) u_1, \ldots, (dR_a) u_k \right) = T_p' \left( (dR_a) u_1, \ldots, (dR_a) u_k \right)$

$= (R_a^* \tau)_{\pi^*} (u_1, \ldots, u_k) = T_p (u_1, \ldots, u_k)$ since $R_{\rho a}^* \tau = \tau_{\rho a}$.

$\Rightarrow \tau$ a smooth: given $x \in M$ choose a local section $s : U \to P$ with $x \in U$. Let $p = s(x), \quad u_i = (ds)_x w_i$.

Then $W_x (w_1, \ldots, w_k) = T_{s(x)} (ds) w_1, \ldots, (ds) w_k) = (s^* \tau)_x (w_1, \ldots, w_k).$
Clearly, \( \omega \) is a smooth 1-form on \( U \in M \). 
\( \Rightarrow \) \( \omega \) is smooth since \( \omega \) is arbitrary, \( \omega \in \Omega^1 \).
This proves \( \Omega^k(P) \text{basic} \in \pi^* \Omega^k(M) \).  

**Proposition 35.2**  
Let \( S^1 \to P \mathrel{\overset{\pi}{\to}} M \) be a principal \( S^1 \) bundle and \( A \in \Omega^1(P, \mathbb{R}) \) a connection 1-form. Then \( dA \in \Omega^2(P) \) is basic hence \( dA = \pi^* F \) for some \( F \in \Omega^2(M) \).

**Proof**  
Since \( S^1 \) is abelian, \( axa^{-1} = x \) for all \( a, b \in S^1 \).
\( \Rightarrow \) \( \text{Ad}(a) = d(ca) = id \) for all \( a \in S^1 \).

Since \( A \) is a connection 1-form, \( R^A_a \) is a \( S^1 \)-invariant vector.
\( \Rightarrow \) \( R^A_a dA = d(R^A_a) = dA + \alpha(a) \).
\( \Rightarrow \) \( dA \) is \( S^1 \)-invariant.

Since \( A \) is \( S^1 \)-invariant, \( \forall x \in \text{Lie}(S^1) \)
\( 0 = L_x A = (d\gamma(x) + \gamma(x)d)A \)
\( = d(A(x)) + \gamma(x) \text{d}A \)

Now \( A(x) = x \), \( \Rightarrow \) \( d(A(x)) = 0 \)
\( \Rightarrow \) \( \gamma(x) \text{d}A = 0 \) for all \( x \in \text{Lie}(S^1) \)
\( \Rightarrow \) \( \forall p \in P, \forall \gamma \in \pi^{-1}(p) \), \( \gamma(u)(dA)_p = 0 \).

\( \therefore \) \( dA \in \Omega^2(P) \) basic.

**Prop 35.1**  
\( dA = \pi^* F \) for some \( F \in \Omega^2(M) \).

**Prop 35.3**  
Let \( A \in \Omega^1(P) \) and \( F \in \Omega^2(M) \) be as above. Then \( F \) is the curvature \( F_A \) of \( A \).

**Proof**  
By definition of curvature, \( \forall x \in M, X, Y \in T_x(M) \)
\( (FA)_x (X(x), Y(x)) = -A_p (L^h_x (Y^h)(p)) \)
for some \( p \in \pi^{-1}(x) \).

Here, as in prev. lecture, \( X^h = \text{horizontal lift of} \ X \)
\( Y^h = \text{vertical lift of} \ Y \).
Recall the 1-form $\alpha$ and a pair of vector fields $V, W$

$$d\alpha(V, W) = V(\alpha(W)) - W(\alpha(V)) - \alpha([V, W]).$$

Hence

$$dA(x^h, y^h) = x^h(A(y^h)) - y^h(A(x^h)) - A([x^h, y^h])$$

which is equal to

$$= 0 - 0 - A([x^h, y^h]).$$

Therefore,

$$dA = \pi^*(FA).$$

Since $\pi^*: \Omega^2(M) \rightarrow \Omega^2(P)$ is injective,

$$FA = F.$$

**Corollary 35.4** The curvature $FA$ of a connection 1-form on a principal $S^1$-bundle is a closed 2-form.

**Proof**

$$\pi^*(dFA) = d(\pi^*FA) = d(dA) = 0.$$ Since $\pi^*$ is injective, $dFA = 0.$

**Prop 35.5** $LF_A \in H^2(M)$ does not depend on $A.$

**Proof** Let $A' \in \Omega^1(P)$ be another connection 1-form.

Let $\alpha = A - A'.$ Then $\forall X \in \text{Lie}(S^1)$

$$\tau(X_P)\alpha = A(X_P) - A'(X_P) = X - X = 0.$$ For $\alpha = \pi^*\beta$ for some $\beta \in \Omega^1(M),$ $\pi^*FA = \pi^*FA' = dA - dA' = d\alpha = d(\pi^*\beta) = \pi^*d\beta.$

Therefore,

$$[FA] = [FA'].$$

□