Last time: Finished proving existence of parallel transport on principal bundles.

Today Curvature

First: general fiber bundles \( F \to Q \xrightarrow{\pi} M \).
\( \mathcal{H} \subset TQ \) horizontal distribution.
Then \( \forall q \in Q \) \( \mathcal{H}_q + V_q = T_q Q \)
and \( \text{ad} \ d\pi_q |_{\mathcal{H}_q} : \mathcal{H}_q \to T_{\pi(q)} M \) is an iso.

For any vector field \( X \) on \( M \) we have its horizontal lift \( \tilde{X}^h \)
\( \tilde{X}^h (q) = (\text{ad} \ d\pi_q |_{\mathcal{H}_q})^{-1} (X (\pi(q))) \).

The curvature of \( \mathcal{H} \) measures how far \( \mathcal{H} \) is from defining a foliation of \( Q \), i.e., a family \( \mathcal{F}_x \) of submanifolds of \( Q \) so that: each \( q \in Q \) is in a unique \( \mathcal{F}_x \) and 
\( T_q (\mathcal{F}_x) = \mathcal{H}_q \).

Ex: \( Q = M \times F \), \( \{ M \times \{ x \} | \ x \in F \} \) is such a family.
For the distribution \( \mathcal{H} = TM \times F \leq T (M \times F) = TM \times TF \)
Note: \( V (M \times F - M) = M \times TF \).

Now, if \( N \subset Q \) is a submanifold, and \( V, W \) are two vector fields on \( N \) tangent to \( N \), then \( [V, W] \) is tangent to \( N \) as well.
So if \( \mathcal{H} \subset TQ \) defines a foliation (which we think of as "curvature of \( \mathcal{H} \) is 0") then any two horizontal lifts \( \tilde{X}^h, \tilde{Y}^h \), we have \( [\tilde{X}^h, \tilde{Y}^h] \) is horizontal again.
Aside 1 \( x^h \) is \( \pi \)-related to \( x \) : \( \pi_1(x^h) = x \cdot \pi \) by def of horizontal lift. \( \Rightarrow [x^h, y^h] \) is \( \pi \)-related to \( [x, y] \). Since horizontal lifts are unique \( [x^h, y^h] \) is horizontal \( \iff \)
\[ [x^h, y^h] = ([x, y])^h \]

So \( [x^h, y^h] - ([x, y])^h \neq 0 \) should mean \( \pi \) has nonzero curvature.

Aside 2 Sections of associated bundles.
Recall if \( G \to P \to M \) is a principal \( G \)-bundle and \( G \to GL(V) \) is a representation, then \( P \times^G V = (P \times V)/G \)
where \( G \) acts on \( P \times V \) by \( a \cdot (p, v) = (p \cdot a^{-1}, p(a)v) \)
If \( f \): \( P \to V \) is any \( C^\infty \) map with
\[ f(p \cdot a^{-1}) = p(a)f(v), \] Then \( f \) defines a map
\[ s_f : P/G \to (P \times V)/G \]
\[ s_f([p]) = [p, f(p)] \]
\[ M = P/G \xrightarrow{s_f} P \times^G V \]

Note! \( s_f \) is smooth.
Reason if \( U \xrightarrow{\sigma} P \) is a local section, then
\[ s_f|U = (s_f \circ \pi)|U = s_\sigma \circ (id \times f) \circ \sigma \], which is \( C^\infty \)
\[ P \xrightarrow{id \times f} P \times V \]
\[ \sigma \downarrow \]
\[ \downarrow n \]
\[ \downarrow s_f \]
\[ U \xrightarrow{s_f} P \times^G V \]
Now let's go back to $G \rightarrow P \rightarrow M$ princ. $G$-bundle, $A \in \Omega^1(P, \mathfrak{g})$ conn 1-form and $\nabla = \ker A$.

Fix two vector fields $X, Y$ on $M$.

Consider the map $f : P \rightarrow \mathfrak{g}$

$$p \mapsto -A_p \left( [X^h, Y^h](p) \right)$$

[We'll see later if we need this - sign.]

Claim $\forall a \in G$, $f(p.a) = Ad(a^{-1}) \cdot f(p)$.

Proof $\forall a \in G$

$$dR_a X^h = X^h \circ R_a, \quad dR_a Y^h = Y^h \circ R_a$$

$$\Rightarrow dR_a \left( [X^h, Y^h] \right) = [X^h, Y^h] \circ R_a$$

$$\Rightarrow f(p.a) = -A_{pa} \left( [X^h, Y^h](pa) \right)$$

$$= -A_{pa} \left( dR_a \left( [X^h, Y^h](p) \right) \right)$$

$$= -(R_a \cdot A)_p \left( [X^h, Y^h](p) \right)$$

$$= -Ad(a^{-1}) \cdot A_p \left( [X^h, Y^h](p) \right) = -Ad(a^{-1}) \cdot f(p)$$

By the above, $f$ defines a section of $P \times \mathfrak{g}$ over $M$.

We get a skew-symmetric bilinear map

$$f_A : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(P \times \mathfrak{g})$$

$$(X, Y) \mapsto \left( \left[ p \mapsto \left[ p \mapsto L_p \cdot -A_p \left( [X^h, Y^h](p) \right) \right] \right)$$

Now, for any $h \in C^\infty(M)$

$$(kY)^h = (k \circ \pi) \cdot Y^h$$

$$\Rightarrow [X^h, (kY)^h] = \left( X^h \cdot (k \circ \pi) \right) \cdot Y^h + \pi \cdot [X^h, Y^h]$$

$$\Rightarrow A \left( L_{X^h}, (kY)^h \right) = (k \circ \pi) \cdot A \left( L_{X^h}, Y^h \right)$$

$$\Rightarrow "f_A"(X, kY) = k \cdot "f_A"(X, Y)$$

Moral $"f_A" \in C^\infty(M)$-bilinear, hence defines a section $F_A$ of $\Lambda^2 T^*M \otimes (P \times \mathfrak{g})$, i.e.
\[ F_A \in \Omega^2 (P \times \mathcal{C}_G). \]

**Def.** \( F_A \) is the curvature of \( A \in \Omega^1 (P, \mathcal{C}_G) \).

Next goal: Given a polynomial \( p \in \mathbb{R} [\mathcal{C}_G] \) which is \( G \)-invariant, \( F_A \) and \( p \) define a class in \( H^2 (M, \mathbb{R}) \) of degree \( 2d \) that doesn't depend on the choice of \( A \).

We think of \( p \) as a symmetric multi-linear map

\[
p : \mathcal{C}_G \times \cdots \times \mathcal{C}_G \rightarrow \mathbb{R}
\]

and \( p \circ F_A \) is then an \( \mathbb{R} \)-value form on \( M \) of degree \( 2d \). This form is closed and its class doesn't depend on \( A \).

Let's first work out a special case: \( G = S^1 \).

Then \( \text{Lie} (S^1) = \mathbb{R} \)

Since \( a, b \in S^1 \) and \( aba^{-1} = b \), \( \text{Ad}(a) = d(ca) = id \)

\[ \text{Ad}(a) = \text{id}_\mathbb{R} \quad \forall a. \]

A principal \( S^1 \) bundle \( P \rightarrow M \), curvature is a form with values in \( P \times S^1 \mathbb{R} = P / S^1 \times \mathbb{R} = M \times \mathbb{R} \)

ie. an ordinary \( 2 \)-form.

Next time: let \( A \in \Omega^1 (P, \mathbb{R}) \) be a connection \( 1 \)-form on an \( S^1 \) principal bundle \( P \rightarrow M \).

Then its curvature \( F_A \) is the unique (closed) \( 2 \)-form defined by

\[ \pi^* F_A = dA. \]