Recall: if $G \rightarrow \mathcal{P} \rightarrow M$ is a principal $G$ bundle and $F$ is a manifold with a left action of $G$, then $P \times^GF := (P \times F)/G \rightarrow M$ is a fiber bundle with typical fiber $F$.

$G$ acts on $P \times F$ by $a \cdot (p, x) = (pa^{-1}, ax)$.

We write $[p, x]$ for the $G$-orbit through $(p, x) \in P \times F$.

If $V$ is a finite dimensional real vector space, and $G$ acts on $V$ by way of a representation $p : G \rightarrow GL(V)$ then $P \times^G V \rightarrow M$ is naturally a vector bundle over $M$ with typical fiber $V$.

If $V$ is complex and $p : G \rightarrow GL(V)$ is a complex representation then $P \times^G V \rightarrow M$ is a complex vector bundle.

Goal for today: A connection 1-form $\omega \in \Omega^1(P, TG)$, equivalently, a $G$-invariant horizontal distribution $\mathcal{H} \subseteq TP$, define a connection $\nabla : \Gamma(TM) \times \Gamma(TE) \rightarrow \Gamma(TE)$ for any associated vector bundle $E = P \times^G V \rightarrow M$.

There are several ways to do this. We'll start with two.

1. $\mathcal{H} \subseteq TP$ defines a $G$-invariant distribution on the fiber bundle $G \times F \rightarrow P \times F \rightarrow M$ ($\mathcal{V}_p(p \times F) = U_p \oplus T_xF$, so $T(p \times F) = \mathcal{H} \oplus (U_p \oplus T_xF)$)

   hence descends to a distribution $\mathcal{H}' \subseteq T(P \times^G F)$:

   $T[p, x](P \times^G F) = \mathcal{H}_x \oplus (U_{p \times \mathcal{H}'} \oplus \text{directions})$

Then, if $E = P \times^G V$, we have a projection

$k : TE \rightarrow \mathcal{H}'$ for $E \in \mathcal{E}$.

We set $(\nabla_S)(m) = k_{sm}(dS_m)$ for $S \in \Gamma(E)$ and $m \in M$. 


It requires a bit of checking to make sure that $\nabla$ is a connection.

(II) For any $x : [0,1] \to M$ we could try to define parallel transport $P_x^E : P_x(a) \to P_x(b)$ $G$-equivariantly. This would induce parallel transport $P_x^E$ on any $E = \mathbb{R}^n \otimes V$ via $F_x(E) = (P_x(t) \times V)/G$. Since $F_x(E) = (P_x(t) \times V)/G$, we would have $P_x^E((1p, v)) = [P_x(t)(1p), v]$. Note that $P_x^E$ is automatically linear since the action of $G$ on $V$ is linear.

Given a section $s \in \Gamma(E)$ and $X \in T_x M$, we get $(\nabla_X s)(m)$ by differentiating $s$ along the curve $\gamma$ with $\gamma(0) = m$, $\gamma'(0) = X$:

$$(\nabla_X s)(m) = \lim_{t \to 0} \frac{1}{t} ((P_x^E)^{-1}(s(\gamma(t)) - s(m)))$$

III. And there is at least one more that uses the fact that any smooth $G$-equivariant function $f : P \to V$ defined a section $s_f : M \to P \times V$ by $s_f(m) = [p, f(p)]$ for any $p \in P_m$.

Note: $[p a^{-1}, f(p a^{-1})] = [p a^{-1}, a f(p)] = [p, f(p)]$ since $f$ is equivariant.

so $s_f$ is well-defined.

What's involved in defining parallel transport on a principal $G$-bundle $G \to P \to M$?

Def. A lift of $\gamma : [0,1] \to M$ to $P$ is a curve $\nu : [0,1] \to P$ with $\pi \circ \nu = \gamma$, i.e., $\nu(t) \in P_{\gamma(t)}$, $\forall t \in [0,1]$.

A horizontal lift of $\gamma : [0,1] \to M$ with respect to a connection $\nabla$ on $P$ (or a 1-form $\alpha \in \Omega^1(P, g)$) is a lift $\nu : [0,1] \to P$ of $\gamma$ with $\nabla_{\nu'}(\nu(t)) = 0$ for $t \in [0,1]$.
It will be convenient to reduce bookkeeping to assume $I_{0,17}$.

Note: Since $\Gamma$ is $G$-invariant, if $u$ is a horizontal lift of $\gamma$ with $w(0) = p$, then for any $a \in G$,
$$w(t) := R_a(u(t)) = u(t)a$$

is a horizontal lift of $\gamma$ with $w(0) = p.a$.

So if we find a horizontal lift of $\gamma$ we get parallel transport
$$P_{\gamma}(t) : P_{\gamma}(0) \to P_{\gamma}(t)$$
by
$$P_{\gamma}(t)(p.a) = u(t)a$$

where $u(t)$ is equivariant by construction.

Note: We can interpret $P_{\gamma}(u)$ as the flow of a vector field on $\gamma^*P \to [0,1]$ and horizontal lifts as integral curves of this vector field. Since $\gamma^*P := \{(t,p) \in (0,1) \times P \mid x(t) = \pi(p)\}$, a lift $w(t)$ of $\gamma(t)$ is a section $t \mapsto (t,w(t))$ of $\gamma^*P$.

Since $d\pi_p : H_p \to T_{x(t)}M$ is an isomorphism, $w(t) \in T_{x(t)}M$ for $\exists! \tilde{w} \in H_p$ with $d\pi_p(\tilde{w}) = w$. So $u(t)$ is a horizontal lift of $\gamma$.

$\tilde{u}(t) \in H_u(t)$ is the unique vector with
$$d\pi(\tilde{u}(t)) = \dot{\gamma}(t), \quad A_{\tilde{u}(t)}(\dot{\gamma}(t)) = 0.$$

Equivalently, since $A^* = d^*A$ is a connection 1-form on $\gamma^*P \to [0,1]$, to define parallel transport on $P$ along $\gamma : [0,1] \to M$, it's enough to integrate the unique horizontal vector field $X$ defined by
$$\int (d\tilde{\sigma})(X) = \frac{d}{dt}, \quad A^*_x(X) = 0$$

Claim: $X(t,p)a = (dR_a)(X(t,p))$, $\forall a \in G$.

Proof: Since $\delta^{\alpha_0} = 0$, $d\tilde{\sigma}(dR_a)(X(t,p)) - d\tilde{\sigma}(X(t,p)) = 0$.

Since $R^*_a A^* = Ad(a^{-1})A^*$, $0 = Ad(a^{-1})A^*_x(X(t,p)) = A^*_x(dR_a(X(t,p)))$.
Corollary. If $\sigma(s)$ is an integral curve of $X$ with $\sigma(0) = (t, p)$, then $R_a(\sigma(s))$ is an integral curve of $X$ as well.

Proof. $\frac{d}{ds} R_a(\sigma(s)) = \frac{d}{ds} X(\sigma(s) \cdot a) = X(\sigma(s) \cdot (R_a(\sigma(s))))$.

Lemma. $\delta^P - [0,1]$ is trivial, i.e., $\delta^P - [0,1]$ has a global section.

Proof. We use the fact if $\sigma, [a,b] \to G$ is $C^\infty$ then for $c > b$, $\sigma$ has an extension $\tilde{\sigma} : [a, c] \to G$ with $\tilde{\sigma}[a,b] = \sigma$. Now, since $\delta^P - [0,1]$ is locally trivial, $\exists n > 0$ s.t. $\forall 0 \leq c < n$,

$X^P], \frac{\ell}{n}, \frac{12}{n}]$ has a section. (Lebesgue lemma). Now, its induction on $n$. Say $X^P], \frac{1}{n}, \frac{12}{n}]$ and $\delta^P], \frac{1}{3}, 1]$ have global sections $\sigma_1$ and $\sigma_2$. Then $g : \left[\frac{1}{3}, \frac{2}{3}\right] \to G$ with $\sigma_2(t) = \sigma_1(t) g(t) + t \in \left[\frac{1}{3}, \frac{2}{3}\right]$. By fact $g$ has an extension $\tilde{g} : \left[\frac{1}{3}, 1\right] \to G$. Define $\sigma(t) = \begin{cases} \sigma_1(t) & 0 \leq t \leq \frac{1}{3} \\ \sigma_2(t) \tilde{g}(t) & \frac{1}{3} \leq t \leq 1 \end{cases}$.

Now, we prove that given a connection $A$ on $P = \pi R \times G \rightleftharpoons \pi R$, the flow of $X \in \Gamma(TP)$ defined by $\frac{d\sigma}{dt}(X) = \frac{d}{dt}, A(\sigma) = 0$, exists for all $s$ with $s \in [0, 1]$ (parallel transport).

Proof. Enough to show: $\forall p \in P$, the flow $\sigma(t, p)$ exists for all $s \in [-\epsilon, \epsilon]$ and all $p \in [0,1] \times G$.

Let $K$ be a compact neighborhood of $t \in G$. Then $[0,1] \times K \subset K$ is compact in $P = \pi R \times G$. $\Rightarrow \exists \varepsilon > 0$ s.t. the integral curve $\psi^P(s)$ of $X$ through $p \in [0,1] \times K$ exists for $s \in [0, \varepsilon]$.

$\left( \psi^P(0) = p, \psi^P(s) = X(\psi^P(0)) \right) \Rightarrow \forall a \in G, \psi^P \cdot a(s) = \psi^P(s) \cdot a$ is the integral curve of $X$ to $\psi^P(s)$ exists for $s \in [0, \varepsilon]$, $\forall p \in [0,1] \times K$. \( \square \)