We are trying to understand connections on principal bundles from two closely related points of view:
- as horizontal distributions
  (i.e. if $G \to P \to M$ is a principal $G$-bundle, a connection is $\mathcal{H}(TP)$ with $\mathcal{H} \oplus UP = TP$)
- as connection 1-forms

Quick aside on exponential maps: if $G$ is a Lie group, $x \in T_1G = \mathfrak{g}$ a vector, we have a left-invariant vector field $\tilde{x}$ defined by $\tilde{x}(a) = (dL_a)_* x$.

One can show (and it is not hard): if $x(t)$ is the integral curve of $\tilde{x}$ with $x(0) = 1_G$, then $x(t+s) = x(t) \cdot x(s)$ for all $t, s$ in $G$.

An integral curve exists for all time.

If $\tilde{x}$ is a 1-parameter subgroup of $G$, i.e. a map of Lie groups $\mathbb{R} \to G$.

Conversely if $\tilde{x}: \mathbb{R} \to G$ is a map of Lie groups, then $x$ is the integral curve of the left-invariant vector field $\tilde{x}$, $\tilde{x}(t) = \dot{x}(0)$.

Things that follow from this:

- if a group $G$ acts on the right on a manifold $M$, then $\forall x \in T_1G$ we have a 1-parameter group of diffeomorphisms $\phi_t: M \to M$ such that $\phi_t(x) = x \cdot (\exp tx)$
- $\phi_{t+s}(x) = x \cdot \exp ((s+t)x) = x \cdot \exp sx \cdot \exp tx = \phi_t(\phi_s(x))$.
\[ X_M(x) = \frac{d}{dt} |_{t=0} (x \cdot \exp tX) \]

**Exercise:** G acts on itself by right mult. What's \( X_G \)?

\[ X_G(x) = \frac{d}{dt} |_{t=0} x \cdot (\exp tX) = \frac{d}{dt} |_{t=0} (L_x (\exp tX)) = (dLx)_x (X) \]

\( \iff X_G \) is the left-invariant vector field defined by \( X \in \mathfrak{t}G \).

ie. \( X_G = \tilde{X} \).

**Exercise:** Recall: if \( a \in G \), \( c_a : G \to G \), \( c_a(x) = axa^{-1} \)

a group homomorphism.

\[ \text{Ad}(a) := (dC_a)_x : T_x G \to T_x G = \text{Id} \]

\[ \text{aside} \quad c_a \circ c_b = c_{ab} \implies \text{Ad}(ab) = \text{Ad}(a) \text{Ad}(b) \]

\[ \implies \text{Ad} : G \to \text{GL}(\mathfrak{g}) \text{ is a homomorphism,} \]

called the Adjoint representation.

\( \iff \forall X \in \mathfrak{g}, \ t \mapsto c_a(\exp tX) \) is a 1-parameter subgroup, so \( c_a(\exp tX) = \exp tY \)

for some \( Y \in \mathfrak{g} \). What is it?

\[ Y = \frac{d}{dt} |_{t=0} c_a(\exp tX) = \frac{d}{dt} |_{t=0} (dC_a)_x (X) = \text{Ad}(a)X \]

Thus

\[ a \exp tX a^{-1} = \exp (t \text{Ad}(a)X) \quad \forall X \in \mathfrak{g} \]

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One more computation: the group \( G \) acts on a manifold \( \mathcal{M} \) on the right, \( X \in \mathfrak{g} \), \( X_\mathcal{M} \) induced vector field.

What's \( (dR_q)_q (X_\mathcal{M}(q)) \) for \( q \in \mathcal{M} \)?
\[
\begin{align*}
(dR_a)_q (X_Q (q)) &= \frac{d}{dt} \bigg|_0 R_a (q \cdot \text{expt} X) \\
&= \frac{d}{dt} \bigg|_0 (q \cdot \text{expt} X) \cdot a = \frac{d}{dt} \bigg|_0 (q.a) (a' \cdot \text{expt} X a) \\
&= \frac{d}{dt} \bigg|_0 R_a(q) \cdot \exp(t \cdot \text{Ad}(a^{-1})X) = (\text{Ad}(a^{-1})X)_q (R_a(t)) \\
\text{Or} \\
\frac{dR_a (X_Q)}{d\tau} &= (\text{Ad}(a^{-1})X)_q \circ R_a \\
&\forall a \in G, \forall X \in g.
\end{align*}
\]

Now back to connection 1-forms, on principal G-bundles.

Recall: \( A \in \Omega^1(P, \mathfrak{g}) \) is a connection 1-form if

1. \( A(X_p (e)) = X \quad \forall X \in \mathfrak{g} \) and
2. \( R_a^* A = \text{Ad}(a^{-1}) \circ A \quad \forall a \in G. \)

Prop. Any principal G-bundle has a conn. 1-form.

Proof. Let \( \pi : P \to M \) be a prin. G-bundle, \( \{U_a\} \) open cover with \( \phi_{ab} : U_{ab} \to P \) partition of unity with \( \text{Supp } \phi_a \subseteq U_a \).

On \( U_a \times G \) we have a connection 1-form: it's the pullback of Maurer-Cartan form on \( G \) to \( U_a \times G \).

Since \( \phi_{ab} \cong U_a \times G \), we have connection 1-forms \( A_a \) on \( \phi_{ab} \).

Now let \( A = \sum \phi_a A_a \).

Proposition (of Spirek, Coursr Intro, Vol 2, p 359) let \( P \to M \) be a principal G-bundle.

1. if \( A \in \Omega^1(P, \mathfrak{g}) \) is a conn 1-form, then \( N = \ker A \) is a horizontal distribution with \( (dR_p)_p N_p = H_{pa} \) \((N \text{ is } G\text{-invariant}) \)

2. Conversely, for any \( G \)-invariant horizontal distribution \( N \) on \( P \), there exists a connection 1-form \( A \in \Omega^1(P, \mathfrak{g}) \) with \( N = \ker A \).
Proof \((\Rightarrow)\) By construction, since \(A_p : U_{P_p} \rightarrow \mathcal{O}_P\),

an iso, \(\ker A_p \oplus U_{P_p} \simeq T_p \mathcal{O}_P\).

Why \((dR_a)_p(\ker A_p) = \ker A_{pa}\)?

We know \(A_p \circ \ker A_p = 0\) = \(A_p(w) = -\text{Ad}(a^{-1})(A_p(w)) = (R_a^*A)_p(w) = \)

\[= A_{pa}(dR_a)_p(w) \Rightarrow (dR_a)(\mathcal{H}_p) \subseteq \mathcal{H}_{pa}, \text{ dimension count } \Rightarrow (dR_a)_p(H_0) = H_{pa}.

(\Leftarrow) Suppose \(T_P = \mathcal{H} \oplus U_P\) and \((dR_a)_p(H_0) = H_{pa}, \forall a \in G\).

Define \(A_p : T_p \rightarrow \mathcal{O}_P\) by

\[A_p(w) = \begin{cases} 0 & \text{if } w \in \mathcal{H}_p \\ X & \text{if } w \in U_{P_p}. \end{cases}\]

Here \(X \in \mathcal{O}_P\) is the unique vector with \((dR_a)(p) = w\).

This define uniquely \(A \in \Omega_1(P, \mathcal{O}_P)\) with \(A_p(X_{\mathcal{O}_P}(p)) = X, \forall X \in \mathcal{O}_P\).

Remains to check: \((R_a^*A)_p = \text{Ad}(a^{-1})A_p \oplus p \in P\).

If \(w \in \mathcal{H}_p\), then \(A_p(w) = 0 \Rightarrow (\text{Ad}(a^{-1})A_p)(w) = 0\).

\((R_a^*A)_p(w) = A_{pa}(dR_a)_p(w) = 0\), since \(dR_a(w) = dR_a\mathcal{H}_p = \mathcal{H}_{pa}\).

If \(w = X_{\mathcal{O}_P}(p)\), then

\[\begin{align*}
(R_a^*A)_p(X_{\mathcal{O}_P}(p)) &= A_{pa}(dR_a)_p X_{\mathcal{O}_P}(p) = \\
&= A_{pa}((\text{Ad}(a^{-1})X)_P(p)) \quad (\text{see 3.3.3}) \\
&= \text{Ad}(a^{-1})X \\
&= \text{Ad}(a^{-1})A_p(X_{\mathcal{O}_P}(p))
\end{align*}\]